Variational Methods in Image Processing for Inpainting and Shadow Removal

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Motivation of our work

Describing and solving, in mathematical words, important and concrete applications.

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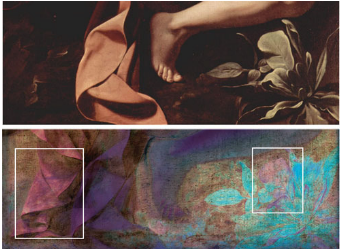
Many research fields have to treat with image processing:

- in cultural heritage;
  - *pentimenti;*



Caravaggio, *John the Baptist* (C. Daffara et al., 2011)

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- in cultural heritage;
  - *pentimenti;*
  - *guide restoration;*

Neidhart von Reuental (C.B. Schönlieb, 2009)

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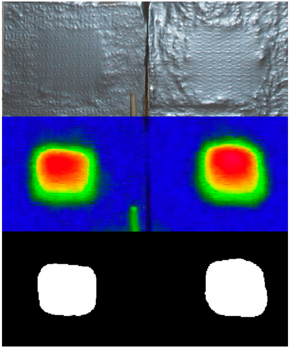
Many research fields have to treat with image processing:

- in cultural heritage;
  - *pentimenti;*
  - *guide restoration;*
  - *thermography;*



Detachments from wall (C. Daffara et al., 2010)

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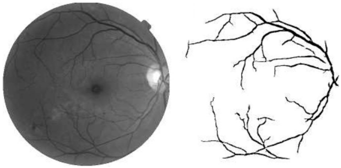

VAMPIRE project (A. Giachetti et al., 2013)

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Describing and solving, in mathematical words, important and concrete applications.

Many research fields have to treat with image processing:

- in cultural heritage;
- in medical imaging;
- in film restoration;

A. Buadès, S. Masnou et al. (2010)

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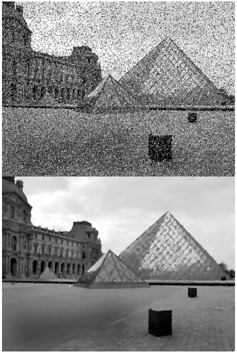
Many research fields have to treat with image processing:

- in cultural heritage;
- in medical imaging;
- in film restoration;
- in image denoising.



A. Chambolle, T. Pock (2010)

Motivation of our work

Describing and solving, in mathematical words, important and concrete applications.

Many research fields have to treat with image processing:

- in cultural heritage;
- in medical imaging;
- in film restoration;
- in image denoising.

In some of these projects I am still involved for further researches.

Plan of our work

This work aims to:

- Study relevant image processing tasks by variational and PDE methods:
  - to model inpainting;
  - to model shadow removal;
- See connection between the 2 problems;
- Show how to speed up the computational time for solving shadow removal;

Mathematical framework:

- Geometric Measure Theory;
- Functions of Bounded Variation;
- Sets of Finite Perimeters;
- Drift-Diffusion equation;

Inpainting - Criminisi et al. (2003)

Shadow Removal - Finlayson et al. (2006)

Basics on BV functions

Key idea: to describe images by gray level lines (all geometric information lie on edges)

- suitable setting: BV functions (natural for describing boundary discontinuities);

Distributional definition of BV function

Let $u :\in \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Then $u \in \mathrm{BV}(\Omega)$ if Du is a vector Radon measure, i.e.

$$\langle Du, \vec{\varphi} \rangle = -\int_\Omega u \operatorname{div} \vec{\varphi} = \int_\Omega \vec{\varphi} \cdot \mathrm{d}Du \quad \forall \vec{\varphi} \in \left[C_0^\infty(\Omega)\right]^m$$

- Du is only concentrated on the boundaries;
- $Du = -\vec{\nu}|Du|$, with $|\nu| = 1$ and $|Du|$-a.e.

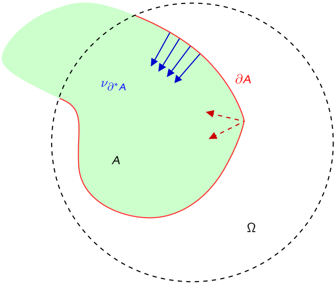
Total variation of u

$$|Du|(\Omega) = \sup \left\{ \int_\Omega u \operatorname{div} \vec{\varphi},\ \|\vec{\varphi}\|_\infty \leq 1,\ \vec{\varphi} \in C_c^\infty(\Omega)^n \right\}$$

- dealing with BV functions $\longrightarrow$ Sets of Finite Perimeters;

Sets of Finite Perimeters

A is of Finite Perimeter in Ω if and only if $\chi_A \in \mathrm{BV}(\Omega)$: $P(A;\Omega) \equiv |D\chi_A|(\Omega)$.



- $D\chi_A$ encodes all geometric information on $\partial A \cap \Omega$;
- $\partial^* A$ (reduced boundary) is $\mathcal{H}^{n-1}$-rectifiable;
- $D\chi_A = \nu_{\partial^* A}\mathcal{H}^{n-1} \llcorner \partial^* A$ so it is concentrated only on the boundaries;
- Gauss-Green: $\int_A \operatorname{div} \varphi = \int_{\partial^* A} \varphi \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1}$;
- measure-theoretic notion of tangent space;

Coarea Formula

Let $u : \mathbb{R}^n \to \mathbb{R}$ a Lipschitz function, $A \subset \mathbb{R}^n$ open.

$$\int_A |\nabla u| = \int_{\mathbb{R}} P(\{u > t\};\, A)\, \mathrm{d}t \quad \text{as elements of } [0, \infty].$$

The total variation of a function is the accumulated surfaces of all its level sets.

Functions of Bounded Variation

- C. Jordan (1881): *functions with control on the oscillations (Fourier series);*
- M. Miranda (1964): $V(u,\Omega) = \sup\left\{ \int_\Omega u \,\mathrm{div}\, \varphi \,\mathrm{d}x : \varphi \in [C_c^1(\Omega)]^n,\ \|\varphi(x)\|_\infty \le 1 \right\}$

The $\mathrm{BV}(\Omega)$ space is a Banach space with the norm $\|u\|_{\mathrm{BV}(\Omega)} = \int_\Omega |u| \,\mathrm{d}x + |Du|(\Omega)$.

- lower semicontinuity: $V(u,\Omega) \le \liminf_{h\to\infty} V(u_h,\Omega)$;
- Convexity: $V(tu_1 + (1-t)u_2,\Omega) \le tV(u_1,\Omega) + (1-t)V(u_2,\Omega)$;
- $u \in [\mathrm{BV}(\Omega)]^m \iff V(u,\Omega) < \infty$;

Theorem (Federer-Vol'pert)

Let $u \in [\mathrm{BV}(\Omega)]^m$. The discontinuity set is $\mathcal{H}^{n-1}$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. Then

$$Du = \underbrace{\nabla u(x)\,\mathrm{d}x}_{D^a u} + \underbrace{(u_+(x) - u_-(x)) \otimes \nu_u(x)\,\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}_{D^j u} + \underbrace{D^s u \llcorner (\Omega \setminus S_u)}_{D^c u}.$$

- $V(u,\Omega) \equiv |Du|(\Omega),\ \forall u \in [\mathrm{BV}(\Omega)]^m$;

Functions of Special Bounded Variation

- introduced by E. De Giorgi, L. Ambrosio (1988);
- good candidate where both volume and surface energies are involved;
- relevant for images;

SBV space

Let $u \in \mathrm{BV}(\Omega)$, then $u \in \mathrm{SBV}(\Omega)$ if $D^c u = 0$:

$$Du = D^a u + D^j u = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u, \quad \forall u \in \mathrm{SBV}(\Omega).$$

$$\mathrm{W}^{1,1}(\Omega) \subset \mathrm{SBV}(\Omega) \subset \mathrm{BV}(\Omega).$$

- if $u \in \mathrm{W}^{1,1}(\Omega)$, or $u \in C^1(\Omega)$, then $Du = D^a u$;
- if $u = \chi_A$ and $|A| < \infty$, then $Du = D^j u$ (not Sobolev because $Du = \nu_A \mathcal{H}^{n-1} \llcorner \partial^* A$);
- if u is the Cantor-Vitali function, then $Du = D^c u$;

Motivational Example

We are involved in several *cutting* and *pasting* domains. Let $u, v \in [\mathrm{BV}(\Omega)]^m$.

- Is $w = u\chi_A + v\chi_{\Omega\setminus A} \in [\mathrm{BV}(\Omega)]^m$?
- Can Dw be expressed?

Let $u, v \in [\mathrm{BV}(\Omega)]^m$, $A \subset \Omega$ a set of finite perimeter with $\partial^* A \cap \Omega$ oriented by ν_A. Let $u^+_{\partial^* A}$, $v^-_{\partial^* A}$ (interior and exterior trace of u and v) given for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* A \cap \Omega$. Then

$$w = u\chi_A + v\chi_{\Omega\setminus A} \in [\mathrm{BV}(\Omega)]^m \iff \int_{\partial^* A\cap\Omega} \left|u^+_{\partial^* A} - v^-_{\partial^* A}\right| \mathrm{d}\mathcal{H}^{n-1} < \infty,$$
$$Dw = Du \llcorner A^1 + (u^+_{\partial^* A} - v^-_{\partial^* A}) \otimes \nu_A \mathcal{H}^{n-1} \llcorner (\partial^* A \cap \Omega) + Dv \llcorner A^0.$$

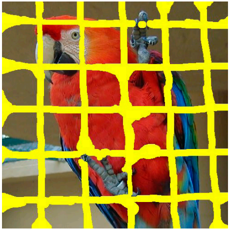
Let $u, v \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $A \subset \Omega$ be a set of finite perimeter. Then

$$w = u\chi_A + v\chi_{\Omega\setminus A} \in \mathrm{SBV}(\Omega),$$
$$Dw = \left[\nabla u\chi_A + \nabla v\chi_{\Omega\setminus A}\right]\mathcal{L}^n + (\widetilde{u} - \widetilde{v})\nu_A \mathcal{H}^{n-1} \llcorner (\Omega \cap \partial^* A).$$

The inpainting problem

- very common in film restoration and image retouching;
- digital inpainting: retouching or recovering damaged ancient paintings (2001);
- we don't want to recover the true missing patch;
- we aim to create a new *natural* one;
- interpolation problem with unknown regularity degree (we are in BV space);
- geometric, sparse or exemplar-based approaches.

Original



Inpainting domain

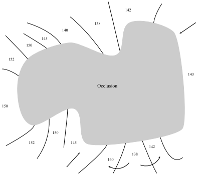
Bornemann (2007)

Geometric approach: The Euler's Elastica

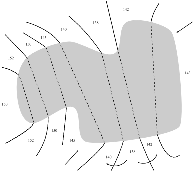
- image smoothness is expressed by *total variation* or *curvature* of level lines;
- the boundary data are propagated to predict the missing geometric structure;
- *local method* based on PDE but fails in presence of texture;
- Γ *Euler's elastica* if it is the equilibrium curve of the elasticity energy (1744):

$$E_2[\gamma] = \int_\gamma (a + b\kappa^2)\,\mathrm{d}s,$$

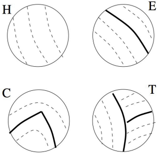
- from C.o.V., we obtain a fourth order equation: $2\kappa''(s) + \kappa^3(s) = \frac{a}{b}\kappa(s)$;



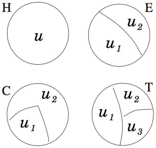
Occlusion



Possible connection



Situation



Approximation

- along any isophote $\gamma_\lambda : u \equiv \lambda$, the curvature of the oriented curve is given by

$$\kappa = \nabla \cdot \vec{n} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) = \mathbf{H} = \kappa_1 + \kappa_2 \text{ (mean curvature)};$$

- $\mathrm{d}t$ is the length element along $\vec{n}$ so $\partial\lambda/\partial t = |\nabla u|$ or $\mathrm{d}\lambda = |\nabla u|\,\mathrm{d}t$;

$$J[u] = E[\mathcal{F}] = \int_0^1 \int_{\gamma_\lambda : u=\lambda} (a + b\kappa^2)\,\mathrm{d}s\,\mathrm{d}\lambda$$
$$\int_0^1 \int_{\gamma_\lambda : u=\lambda} \left(a + b\left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right) |\nabla u|\,\mathrm{d}t\,\mathrm{d}s$$
$$\int_D \left(a + b\left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right) |\nabla u|\,\mathrm{d}x, \text{ with } u \in \mathrm{BV}(\Omega).$$

- suitable boundary conditions;
- if $a/b = \infty$, then $TV(u) = \int_\Omega |\nabla u|$, with the condition $u|_{\Omega \setminus D} = u_0|_{\Omega \setminus D}$.

Theorem: The noise free TV inpainting model (Chan, Shen)

Suppose that $u_0 \in \mathrm{BV}(\Omega)$, $u_0 \subset [0, 1]$. Then the *noise free TV inpainting model* $TV(u)$, together with the gray value constraint $u \subset [0, 1]$, has one optimal inpainting at least.

Sparse approach based on (consistent) dictionaries

Input

Sparse approach based on (consistent) dictionaries

Hays - Efros

Exemplar-based approach: Variational Framework

- patches similarity within the image: correspondence maps $\varphi : O \to O^c$;
- the whole image is scanned (greedy algorithm but sensitive to the order);
- Demanet (2003): variational formulation searching $u(x) = \widehat{u}(\varphi(x))$, for $x \in O$:

$$E(\varphi) = \int_O \int_{\Omega_p} \left|\widehat{u}(\varphi(x+h)) - \widehat{u}(\varphi(x)+h)\right|^2 \mathrm{d}h\, \mathrm{d}x \quad \text{(non-convex);}$$

- Gilboa, Osher (2007): replace φ with weights $w(x,y)$, subject to $\int_{\widetilde{O}^c} w(x,y) = 1$;

Arias, Caselles, Facciolo (2011)

$$\min \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y)\varepsilon\left(p_u(x) - p_{\widehat{u}}(y)\right) \mathrm{d}y\, \mathrm{d}x + T \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) \log(w(x,y))\, \mathrm{d}y\, \mathrm{d}x$$

- when $T \to 0$ the weights are the correspondence map: $w(x,\widehat{x}) = \delta(\widehat{x} - \varphi(x))$.
- NL Means: $\mathbb{P} \equiv \mathrm{L}^2(\Omega_p)$, $\varepsilon(p_u(x) - p_{\widehat{u}}(y)) = \|p_u(x) - p_{\widehat{u}}(y)\|_g^2$.
- NL Poisson: $\mathbb{P} \equiv \mathrm{W}^{1,2}(\Omega_p)$, $\varepsilon(p_u(x) - p_{\widehat{u}}(y)) = \|p_u(x) - p_{\widehat{u}}(y)\|_{\nabla,g}^2$.
- NL Gradient Medians: $\mathbb{P} \equiv \mathrm{BV}(\Omega_p)$, $\varepsilon(p_u(x) - p_{\widehat{u}}(y)) = \|p_u(x) - p_{\widehat{u}}(y)\|_{\nabla,g}$.

- solution's structure: rototranslation of patches;

NL-Poisson patch metric function

$$\mathcal{E}_{\nabla,T}(u,w) = \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) \| p_u(x) - p_{\widehat{u}} \|^2_{g,\nabla} \, \mathrm{d}y \, \mathrm{d}x + T \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) \log w(x,y) \, \mathrm{d}y,$$

Euler-Lagrange equations respect w and u ▸ Osmosis

$$w_{\varepsilon,T}(u)(x,y) = \frac{1}{Z_{\varepsilon,T}(u)(x)} \exp\left(-\frac{1}{T} \varepsilon\big(p_u(x) - p_{\widehat{u}}(y)\big) \right),$$

$$\begin{cases} \Delta u(z) = \operatorname{div} \mathbf{v}(w)(z), & z \in O, \\ u = \widehat{u}, & \text{in } \partial O, \end{cases} \implies \min \int_{\widetilde{O}} \| \nabla u(z) - \mathbf{v}(w)(z) \|_2^2 \, \mathrm{d}z.$$

Existence of minima for NL-Means and NL-Poisson

Existence of minima for NL-Means approach - Arias et al. (2011)

Assume $g \in C_c(\mathbb{R}^n)^+$, supp $g \in \Omega_p$, $\nabla g \in \mathrm{L}^1(\mathbb{R}^n)$ and $\widehat{u} \in \mathrm{BV}(O^c) \cap \mathrm{L}^\infty(O^c)$.

- If $(u_n, w_n) \in \mathcal{A}_2$ is a minimizing sequence for $\mathcal{E}_{2,T}$ such that u_n is uniformly bounded, then we may extract a subsequence converging to a minimum of $\mathcal{E}_{2,T}$.
- There exist a minimum $(u, w) \in \mathcal{A}_2$ of $\mathcal{E}_{2,T}$. For any minimum $(u, w) \in \mathcal{A}_2$ we have that $u \in W^{1,\infty}(O)$ and $w \in W^{1,\infty}(\widetilde{O} \times \widetilde{O}^c)$.

Existence of minima for NL-Poisson approach - Arias et al. (2011)

Assume $\widehat{u} \in \mathrm{W}^{2,2}(O^c) \cap \mathrm{L}^\infty(O^c)$, $g \in \mathrm{W}^{1,\infty}(\mathbb{R}^n)^+$, supp $g \in \Omega_p$ compact.

- There exists a solution of the variational problem $\min_{(u,w)\in\mathcal{A}_\nabla} \mathcal{E}_{\nabla,T}(u, w)$.
- For any solution $(u, w) \in \mathcal{A}_\nabla$ we have $u \in \mathrm{W}^{1,2}(O) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(O) \cap \mathrm{L}^\infty(O)$ for all $p \in [1, \infty]$ and $w \in \mathrm{W}^{1,\infty}(\widetilde{O} \times \widetilde{O}^c)$.

Algorithms and Visual Results - Arias, Caselles, Facciolo (2011)

Some numerical details:

- Patchmatch (2009) for patch comparison: faster than kd-tree;
- because of high probability to fall in local minima: multiscale approach;

Alternating optimization for NL-means model

Input: u^0 with $\|u^0\|_\infty \le \|\widehat{u}\|_\infty$.
for each $k \in \mathbb{N}$ **do**
  $w^{k+1} = \arg\min_{w \in \mathcal{W}} \mathcal{E}_{2,T}(u^k, w)$,
  $u^{k+1} = \arg\min_{u} \mathcal{E}_{2,T}(u, w^{k+1})$.
end for

Alternating optimization for NL-Poisson model

Input: u^0 with $\|u^0\|_\infty \le \|\widehat{u}\|_\infty$.
for each $k \in \mathbb{N}$ **do**
  $w^{k+1} = \arg\min_{w \in \mathcal{W}} \mathcal{E}_{\nabla,T}(u^k, w)$,
  $u^{k+1} = \arg\min_{u \in W^{1,2},\, u|_{\partial O^c} = \hat{u}|_{\partial O^c}} \mathcal{E}_{\nabla,T}(u, w^{k+1})$.
end for

Original | KSY | NL-Means | NL-Poisson

Drift Diffusion PDE - Weickert (2013)

- Osmosis: omnipresent in nature (it transports water across membranes);
- diffusion (symmetric processes) leads to flat steady states;
- osmosis (nonsymmetric counterpart of diffusion) allows nonconstant steady states;
- a system is in a steady state for a property p if $\partial_t p = 0$.
- Fokker-Plank equation (time evolution of the p.d.f. of the velocity of a particle)

The continuous model

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \operatorname{div}(\mathbf{d}u), & \text{on } \Omega \times (0,T] \\ u(\mathbf{x},0) = f(\mathbf{x}), & \text{on } \Omega \\ \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0,T] \end{cases}$$

- preservation of the Average Grey Value;
- preservation of Positivity;
- convergence to Nontrivial Steady State when $\mathbf{d} = \nabla \log v$;

Associated minimization problem NL-Poisson

$$\min \int_\Omega v \left| \nabla \left(\frac{u}{v} \right) \right|^2 \mathrm{d}x, \text{ or } \min \int_\Omega |\nabla u - \mathbf{d}u|^2 \, \mathrm{d}x$$

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The discrete model

$$
\begin{aligned}
\frac{\partial u_{i,j}}{\partial t} = &\left(\frac{1}{h^2} - \frac{d_{1,i+\frac{1}{2},j}}{2h}\right) u_{i+1,j} + \left(\frac{1}{h^2} + \frac{d_{1,i-\frac{1}{2},j}}{2h}\right) u_{i-1,j} \\
&+ \left(\frac{1}{h^2} - \frac{d_{2,i,j+\frac{1}{2}}}{2h}\right) u_{i,j+1} + \left(\frac{1}{h^2} + \frac{d_{2,i,j-\frac{1}{2}}}{2h}\right) u_{i,j-1} \\
&+ \left(-\frac{4}{h^2} - \frac{d_{1,i+\frac{1}{2},j}}{2h} + \frac{d_{1,i-\frac{1}{2},j}}{2h} - \frac{d_{2,i,j+\frac{1}{2}}}{2h} + \frac{d_{2,i,j-\frac{1}{2}}}{2h}\right) u_{i,j} = P[u_{i,j}].
\end{aligned}
$$

- suppose to know shadow boundaries for applications we have in mind;

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \operatorname{div}(\mathbf{d}u) \implies \partial_t u = Pu \text{ (only } P \text{ needed)}$$

- Exponential Integrators;

$$\begin{cases} \mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{b}(t, \mathbf{y}(t)), & t > t_0 \\ \mathbf{y}(t_0) = \mathbf{y}_0, \end{cases}$$

whose analytical solution is, with $\varphi_1(z) = (e^z - 1)/z$

$$\mathbf{y}(t) = \exp((t - t_0)A)\mathbf{y}_0 + \int_{t_0}^{t} \exp((t-\tau)A)\mathbf{b}(\tau, \mathbf{y}(\tau))\, d\tau,$$

$$\mathbf{y}(t) = \exp((t - t_0)A)\mathbf{y}_0 + (t - t_0)\varphi_1((t - t_0)A)\mathbf{b} = \mathbf{y}_0 + (t - t_0)\varphi_1((t - t_0)A)(A\mathbf{y}_0 + \mathbf{b}).$$

- no need to compute $\exp(P)$ but $\exp(P)u$ (Krylov methods for P big and sparse)

$$A = V_m H_m V_m^T \implies \exp(A)V_m \approx V_m \exp(H_m) \implies \exp(A)v \approx V_m \exp(H_m)e_1$$

- Euler exponential method is exact if $b(\mathbf{y}(t)) = b(\mathbf{y}_0) \equiv \mathbf{b}$ or of order 1 otherwise.
- scripts from *Al-Mohy and Higham(2011)* and *Sidje(1998)* have been tested;

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t$$

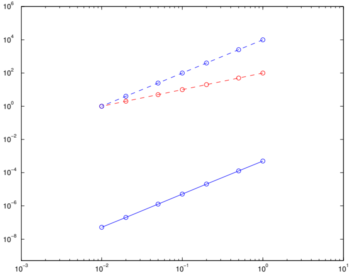
- Exponential Integrators;
- θ-method with direct method: LUpq factorization

```
[L,U,p,q]=lu(I-dt*theta*A,'vector');
B = (I+dt*(1-theta)*A);
for t = (dt:dt:T)
  C = B*y;
  y(q) = U\(L\(noto(p)));
end
```

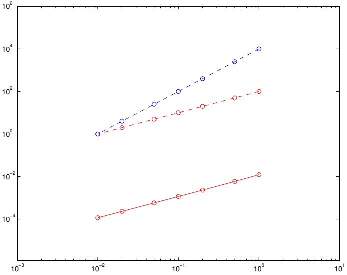
Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization



LUpq, $\theta = 0.5$



LUpq, $\theta = 1$

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \operatorname{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
  - BiCGStab: standard, fixed timestep dt until T fixed is reached;

```
y=bicgstab(I-dt*theta*A,(I+dt*(1-theta)*A)*y,tol,maxit);
y=bicgstab(I-dt*theta*A,(I+dt*(1-theta)*A)*y,tol,maxit,[],[],y);
y=bicgstab(I-dt*theta*A,(I+dt*(1-theta)*A)*y,tol,maxit,L,U,y);
```

It is not satisfactory at all (first steps are the most important ones - far away from steady state).

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
  - BiCGStab: standard, fixed timestep dt until T fixed is reached;
  - A-BiCGStab: adaptative, variable timestep dt until T fixed is reached;

```
% k = number of iterations in BiCGStab
averit = 35*maxit/50;
safe_zone = [0.8*averit,1.2*averit];
```

$$\texttt{dt(t+1)} = \begin{cases} \texttt{1.2*dt(t)} & \texttt{if k<min(safe\_zone) steps} \\ \texttt{1.0*dt(t)} & \texttt{if min(safe\_zone)<k<max(safe\_zone) steps} \\ \texttt{0.8*dt(t)} & \texttt{if k>max(safe\_zone) steps} \\ \texttt{0.5*dt(t)} & \texttt{otherwise (don't increase t).} \end{cases}$$

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
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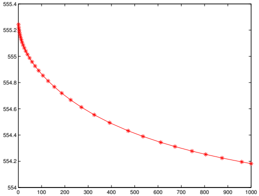
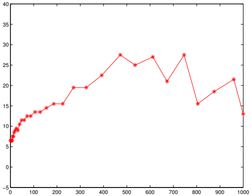
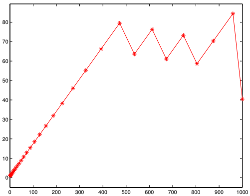


A-BiCGStab: `norm(y)`



A-BiCGStab: Iters

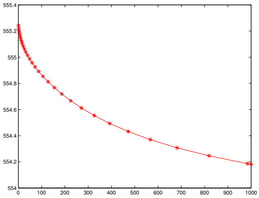


A-BiCGStab: dt

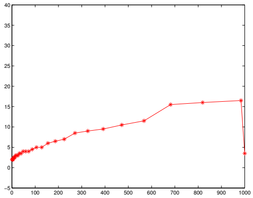
Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{d}t\theta P)^{-1}(I + \mathrm{d}t(1-\theta)P)u^t$$

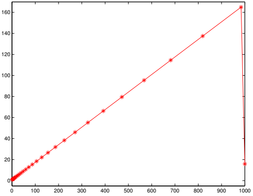
- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
  - BiCGStab: standard, fixed timestep $\mathrm{d}t$ until T fixed is reached;
  - A-BiCGStab: adaptative, variable timestep $\mathrm{d}t$ until T fixed is reached;



A-BiCGStab+ilu: `norm(y)`



A-BiCGStab+ilu: Iters



A-BiCGStab+ilu: $\mathrm{d}t$

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
  - BiCGStab: standard, fixed timestep dt until T fixed is reached;
  - A-BiCGStab: adaptative, variable timestep dt until T fixed is reached;
  - F-BiCGStab: standard, fixed timestep dt until exit condition is true;

```
norm(y_new-y)/norm(y_new) < dt * tol_exit;
```

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t$$

- Exponential Integrators;
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  - BiCGStab: standard, fixed timestep dt until T fixed is reached;
  - A-BiCGStab: adaptative, variable timestep dt until T fixed is reached;
  - F-BiCGStab: standard, fixed timestep dt until exit condition is true;
  - FA-BiCGStab: adaptative, variable timestep dt until exit condition is true;

```
norm(y_new-y)/norm(y_new) < dt(t) * tol_exit;
```

Solving the Drift-Diffusion PDE for Shadow Removal

$$u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t;$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
- semi-Fourier collocation, full-Fourier collocation;
  - mirror the image to guess the periodic boundary condition;
  - large timestep: error from reference is upper bounded by Gibbs phenomenon on high jumps of colours;
  - **d** can be modified when/where necessary (e.g. $\mathbf{d} = \mathbf{d}. * u_{\text{mask}}$);

Solving the Drift-Diffusion PDE for Shadow Removal

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- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
- semi-Fourier collocation, full-Fourier collocation;

$$(I - \mathrm{dt}\theta_1\Delta - \mathrm{dt}\theta_2 D)u^{t+1} = (I + \mathrm{dt}(1-\theta_1)\Delta + \mathrm{dt}(1-\theta_2)D)u^t;$$

Algorithm 1: Semi-implicit solver with bridge Fourier collocation

Input: u^0 (original image, $2\mathrm{D}$ matrix of N rows and M columns), k pixel-indexes.
Output: u^T at time $T = t^{\mathrm{end}}$.
1: coeff $= (1 + 4k\pi^2\mathrm{dt})$;
2: $A = (I + \mathrm{dt}D)$;
3: **for** $t = \mathrm{dt} : \mathrm{dt} : T$ **do**
4: $u = \mathrm{reshape}(Au(:), N, M)$; $\hat{v} = \mathrm{fft2}(u)./\mathrm{coeff}$;
5: $u = \mathrm{ifft2}(\hat{v})$;
6: **end for**

Solving the Drift-Diffusion PDE for Shadow Removal

$$u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t;$$

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5: $\quad u = \mathrm{ifft2}(\hat{v})$;
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Solving the Drift-Diffusion PDE for Shadow Removal

$$u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t;$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
- semi-Fourier collocation, full-Fourier collocation;

$$u = \sum_{|k|_\infty \le N} u_k \mathrm{e}^{2\mathrm{i}\pi x\cdot k} \quad \text{and} \quad \Delta u = -\sum_{|k|_\infty \le N} u_k 4\pi^2 |k|^2 \mathrm{e}^{2i\pi x\cdot k};$$

Algorithm 1: Semi-implicit solver with bridge Fourier collocation

Input: u^0 (original image, $2\mathrm{D}$ matrix of N rows and M columns), k pixel-indexes.
Output: u^T at time $T = t^{\text{end}}$.

1: coeff $= (1 + 4k\pi^2\mathrm{dt})$;
2: $A = (I + \mathrm{dt}D)$;
3: **for** $t = \mathrm{dt} : \mathrm{dt} : T$ **do**
4: $\quad u = \text{reshape}(Au(:), N, M)$; $\hat{v} = \text{fft2}(u)./\text{coeff}$;
5: $\quad u = \text{ifft2}(\hat{v})$;
6: **end for**

Solving the Drift-Diffusion PDE for Shadow Removal

$$u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t;$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
- semi-Fourier collocation, full-Fourier collocation;

$$\sum_{|k|_\infty \leq N} v_k \mathrm{e}^{2\mathrm{i}\pi x\cdot k} = v = (I - \mathrm{dt}\Delta)u = \sum_{|k|_\infty \leq N} (1 + \mathrm{dt}4\pi^2|k|^2)u_k \mathrm{e}^{2\mathrm{i}\pi x\cdot k};$$

Algorithm 1: Semi-implicit solver with bridge Fourier collocation

Input: u^0 (original image, $2\mathrm{D}$ matrix of N rows and M columns), k pixel-indexes.
Output: u^T at time $T = t^{\mathrm{end}}$.

1: coeff $= (1 + 4k\pi^2\mathrm{dt})$;
2: $A = (I + \mathrm{dt}D)$;
3: **for** $t = \mathrm{dt} : \mathrm{dt} : T$ **do**
4: u = reshape$(Au(:), N, M)$; $\hat{v}$ = fft2(u)./coeff;
5: u = ifft2$(\hat{v})$;
6: **end for**

Solving the Drift-Diffusion PDE for Shadow Removal

$$u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t;$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
- semi-Fourier collocation, full-Fourier collocation;

$$u_k = \frac{v_k}{(1+\mathrm{dt}4\pi^2|k|^2)}.$$

Algorithm 1: Semi-implicit solver with bridge Fourier collocation

Input: u^0 (original image, $2\mathrm{D}$ matrix of N rows and M columns), k pixel-indexes.
Output: u^T at time $T = t^{\mathrm{end}}$.

1: coeff $= (1 + 4k\pi^2\mathrm{dt})$;
2: $A = (I + \mathrm{dt}D)$;
3: **for** $t = \mathrm{dt} : \mathrm{dt} : T$ **do**
4: $\quad u = \mathrm{reshape}(Au(:), N, M)$; $\hat{v} = \mathrm{fft2}(u)./\mathrm{coeff}$;
5: $\quad u = \mathrm{ifft2}(\hat{v})$;
6: **end for**

Solving the Drift-Diffusion PDE for Shadow Removal

$$\partial_t u = \Delta u - \mathrm{div}(\mathbf{d}u) \implies u^{t+1} - \mathrm{dt}\Delta u^{t+1} = u^t - \mathrm{dt}\,\mathrm{div}((\nabla \log u)\, u^t);$$

- Exponential Integrators;
- θ-method with direct method: LUpq factorization;
- θ-method with iterative method: BiCGStab and variants;
- semi-Fourier collocation, full-Fourier collocation;

Algorithm 2: Semi-implicit solver with fully Fourier collocation

Input: u^0 (original image, $2\mathrm{D}$ matrix of N rows and M columns), k pixel-indexes.
Output: u^T at time $T = t^{\text{end}}$.

define the `flag_log` $= \{0, 1\}$ variable, useful to change the computation of $\mathbf{d}$.
if `flag_log` **then**
  $\mathbf{d} = \nabla \log u = \mathrm{ifft2}(\mathrm{fft2}(\log u).*(2\pi \mathrm{i} k))$
else
  $\mathbf{d} = \nabla u./u = (\mathrm{ifft2}(\mathrm{fft2}(u).*(2\pi \mathrm{i} k)))./u;$
end if
$\mathrm{coeff} = (1 - 4k\pi^2 \mathrm{dt});$
for $t = \mathrm{dt} : \mathrm{dt} : T$ **do**
  $\mathrm{div}(\mathbf{d}u) = \mathrm{ifft2}((\mathrm{fft2}(\mathbf{d}u).*(2\pi \mathrm{i} k)));\quad u = \mathrm{ifft2}(\mathrm{fft2}(u - \mathrm{dt}\,\mathrm{div}(\mathbf{d}u))./\mathrm{coeff})$
end for

Numerical Results

Parameters: dt = 1 (*for Fourier (F), dt = 100), `tol_bicgstab`=10^{-06}, `tol_exit`=10^{-06} and maxit = 30.

θ = 0.5	**LUpq**	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	5963	7177.26	1000	1000	6933	7579.16
I	-	2968.5	1359	15394	4082	1585	516	9979.5	1485
R	-	0	0	0	4	0	0	0	2
E	1.47e-08	1.68e-04	1.19e-03	-	-	1.03e-04	1.21e-03	-	-
C	86.80	138.12	24.43	804.32	73.34	192.73	19.37	1308.37	51.36

θ = 1	**LUpq**	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	5618	7798.39	1000	1000	6935	7856.53
I	-	3306.5	1717	15897.5	4529.5	1785	753	10180.5	2298.5
R	-	0	1	0	13	0	0	0	2
E	4.59e-05	1.47e-04	4.53e-03	-	-	2.04e-04	3.94e-03	-	-
C	84.27	125.36	30.04	679.08	83.84	181.92	25.63	1203.16	75.19

	Ref. `expmv`	**LUpq**	**BiCGStab**	**BiCGStab + ilu**	`expv`	**F. Alg. 1**	**F. Alg. 2**
T	1000	1000	1000	1000	1000	1000*	1000*
θ	-	1	0.5	0.5	-	-	-
I	-	-	1359	516	-	-	-
R	-	-	0	0	-	-	-
E	-	4.59e-05	1.19e-03	1.21e-03	1.73e-04	0.1275	0.1080
C	206.26	84.27	24.43	19.37	25.07	4.80	10.94

Numerical Results

Parameters: $\mathrm{dt} = 1$ (*for Fourier (F), $\mathrm{dt} = 100$), `tol_bicgstab`=10^{-07}, `tol_exit`=10^{-07} and $\mathrm{maxit} = 30$.

$\theta = 0.5$	**LUpq**	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	16619	23061.54	1000	1000	20430	24228.54
I	-	3398.5	1872	38248	11660	3066	680.5	27837.5	3859
R	-	0	4	0	24	0	0	0	7
E	1.47e-08	2.24e-05	4.09e-03	-	-	9.58e-06	1.20e-03	-	-
C	86.80	147.61	35.2	1996	215.72	233.46	24.33	3428.48	130.35

$\theta = 1$	**LUpq**	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	15358	22299.95	1000	1000	20433	21338.26
I	-	4032.5	2407.5	39377.5	10806	3332.5	1036	31189	5774.5
R	-	0	4	0	47	0	0	0	6
E	4.59e-05	6.59e-05	4.80e-03	-	-	7.21e-05	3.22e-03	-	-
C	84.27	137.13	43.44	1748.14	209.73	225.12	34.25	3240.42	187.02

	Ref. `expmv`	**LUpq**	**BiCGStab**	**BiCGStab + ilu**	`expv`	**F. Alg. 1**	**F. Alg. 2**
T	1000	1000	1000	1000	1000	1000*	1000*
θ	-	1	0.5	0.5	-	-	-
I	-	-	1872	680.5	-	-	-
R	-	-	4	0	-	-	-
E	-	4.59e-05	4.09e-03	1.20e-03	1.73e-04	0.1275	0.1080
C	206.26	84.27	35.2	24.33	25.07	4.80	10.94

Input

Reference with `expmv.m`

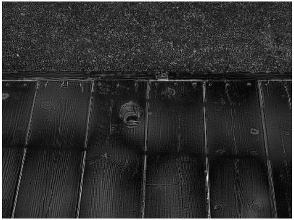
LUpq, $\theta = 1$

A-BiCGStab + ilu, $\theta = 0.5$

Input

Reference with `expmv.m`

Fourier Alg. 1

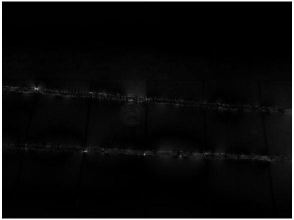


Error Alg. 1

Input

Reference with `expmv.m`

Fourier Alg. 2 with $\mathbf{d} = \nabla u/u$

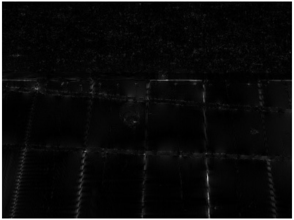


Error Alg. 2 with $\mathbf{d} = \nabla u/u$

Input

Reference with `expmv.m`

Fourier Alg. 2 with $\mathbf{d} = \nabla \log u$



Error Alg. 2 with $\mathbf{d} = \nabla \log u$

Input

F-BiCGStab, $\theta = 0.5$

F-BiCGStab + ilu, $\theta = 0.5$

Input

FA-BiCGStab, $\theta = 0.5$

FA-BiCGStab + ilu, $\theta = 0.5$

Other application: seamless image cloning

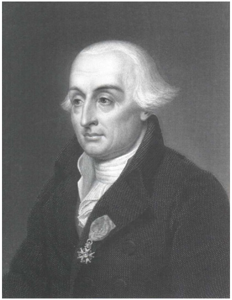
- fuse incompatible information - *Poisson Image Editing*, Perez (2003);
- interpolant f_2 of f_1 over Γ is the solution of

$$\text{(Euler - Lagrange)}\ \Delta f_2 = 0,\ \text{on } \Gamma,\ \text{with } f_2 = f_1,\ \text{on } \partial\Gamma \implies \text{blurred};$$

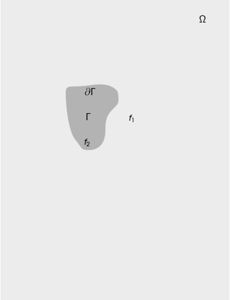
- *guidance vector field* **p**:

$$\text{(Euler - Lagrange)}\ \Delta f_2 = \operatorname{div}\mathbf{p},\ \text{on } \Gamma,\ \text{with } f_2 = f_1,\ \text{on } \partial\Gamma \implies p = \nabla f_2;$$

Euler



Lagrange



Notation

Input

Other application: seamless image cloning

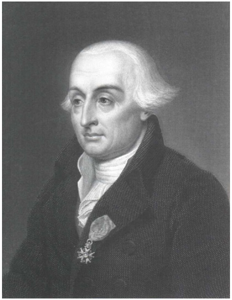
- fuse incompatible information - *Poisson Image Editing*, Perez (2003);
- interpolant f_2 of f_1 over Γ is the solution of

$$\text{(Euler - Lagrange) } \Delta f_2 = 0, \text{ on } \Gamma, \text{ with } f_2 = f_1, \text{ on } \partial\Gamma \implies \text{blurred};$$

- *guidance vector field* **p**:

$$\text{(Euler - Lagrange) } \Delta f_2 = \operatorname{div} \mathbf{p}, \text{ on } \Gamma, \text{ with } f_2 = f_1, \text{ on } \partial\Gamma \implies p = \nabla f_2;$$

Euler



Lagrange

Pérez (2003)

Osmosis: mean on $\partial\Gamma$

Conclusions and Future Works

- Fourier is the fastest way tested despite of a visually negligible Gibbs phenomenon;
- FA-BiCGStab or Exponential Integrators are alternative approaches;
- connection between *NL-Poisson inpainting* and *shadow removal* problems;
- better control on stopping criterion for BiCGStab;
- to model non-constant shadow areas: variational model to *inpaint* the light?
- simple old equations are still useful to model new computer vision problems;

Thank you for your attention.