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Variational methods in Image Processing for Inpainting and Shadow Removal

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Treat nature in terms of the cylinder, the sphere, and the cone, the whole put into perspective so that each side of an object, or of a plane, leads towards a central point.

— Paul Cézanne

Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

— Benoît Mandelbrot

Abstract

This work deals with variational and PDE methods for computer vision applications such as *inpainting*, *shadow removal* and other image processing tasks. Inpainting (as well as image restoration) is a wide and open research field: when a corrupted image is given, typically with a hole where data are missing, one wants to get a credible retouching based on the information surrounding the hole itself. This problem can be attacked with geometric or exemplar-based methods: the former consider the propagation of level lines according to the curvature κ, as e.g. [Masnou and Morel(1998), Chan *et al.*(2002)]; the latter, presented for instance in the very recent work [Arias *et al.*(2011)], extends the texture-oriented approach by [Efros and Leung(1999)] minimizing a functional whose core is a patch-comparison metric distance. One of these metrics, called *Non local Poisson*, is associated to a diffusion-transport Euler-Lagrange equation and provides good results when the illumination changes within the inpainting domain: the same equation arises in contexts dealing with lightings, such as the Shadow Removal problem, where the aim is to recover the information underlying shadow areas. The Shadow Removal problem can be modelled as in [Weickert *et al.*(2013)] by means of a drift-diffusion equation, and solved, as in [Vogel *et al.*(2013)] with the iterative BiCGStab solver. In our work, we aimed to test other numerical methods such as Exponential Integrators, which are exact in the time discretization domain, and Fourier-based collocation methods. For the sake of completeness, we propose here also some variants to the standard BiCGStab algorithm in order to adapt the timesteps to the number of iterations expected from the iterative solver. This is useful when the precision of the solution must be seriously considered while Exponential Integrators result too much slow. Our contribution shows a very strong speed up in the computation time (despite of a visually negligible Gibbs phenomenon when using the Fourier approach).

This thesis is organized as follows: the first three chapters deal with the necessary theoretical background concerning basic notions and main results of Geometric Measure Theory, Finite Perimeter Sets and Functions of Bounded Variation (BV for short). In Chapter 4, we discuss the Inpainting Problem in a BV context with both previously cited approaches. Finally, in Chapter 5, we describe the connection between the Inpainting and the Shadow Removal Problem by completing our dissertation with some numerical experiments.

Contents

CHAPTER 1

Measure Theory

We introduce some results from Measure Theory and Functional Analysis. For more details, we refer to [Brezis(2010)] and [Ambrosio *et al.*(2000)].

1.1 Basic results

Definition 1.1 (σ-algebra and measure spaces)**.** Let X be a non-empty set and let $\mathscr{A} \subset \mathscr{P}(X)$ a collection of subsets of X. Then

- $\mathscr{A}$ is an *algebra* if $\emptyset \in \mathscr{A}$, $A_1 \cup A_2 \in \mathscr{A}$ and $X \setminus A_1 \in \mathscr{A}$ whenever $A_1, A_2 \in \mathscr{A}$;
- an algebra $\mathscr{A}$ is a σ*-algebra* if for any sequence $\{A_h\}_{h\in\mathbb{N}} \subset \mathscr{A}$ we have $\bigcup_{h\in\mathbb{N}} A_h \in \mathscr{A}$;
- if $\mathscr{A}$ is a σ-algebra in X, we call the pair $(X, \mathscr{A})$ a *measure space*.

Definition 1.2 (Positive measure)**.** Let $(X, \mathscr{A})$ be a measure space and $\mu : \mathscr{A} \to [0, \infty]$. Then

- μ is *additive* if, for $A_1, A_2 \in \mathscr{A}$,

$$A_1 \cap A_2 = \emptyset \implies \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2);$$

- μ is σ*-subadditive* if, for $A \in \mathscr{A}$ and $\{A_h\}_{h\in\mathbb{N}} \subset \mathscr{A}$,

$$A \subset \bigcup_{h=0}^{\infty} A_h \implies \mu(A) \leq \sum_{h=0}^{\infty} \mu(A_h);$$

- μ is a *positive measure* if $\mu(\emptyset) = 0$ and μ is σ-additive on $\mathscr{A}$, i.e. for any sequence $\{A_h\}_{h\in\mathbb{N}}$ of pairwise disjoint elements of $\mathscr{A}$ we have

$$\mu\left(\bigcup_{h=0}^{\infty} A_h\right) = \sum_{h=0}^{\infty} \mu(A_h);$$

- μ is *finite* if $\mu(X) < \infty$;
- the set $A \subset X$ is *σ-finite* with respect to positive measure μ if it is the union of an increasing sequence of sets with finite measure. If X itself is σ-finite, then μ is *σ-finite*;
- a positive measure μ such that $\mu(X) = 1$ is also called a *probability measure*.

We introduce the *vector-valued measures*, which is a key definition in the BV function theory because the gradient of a Bounded Variation function, in the sense of distributions, is a measure of this kind.

Definition 1.3 (Real and vector measure)**.** Let $(X, \mathscr{A})$ a measure space and $m \in \mathbb{N}$, $m \geq 1$. Then

- $\mu : \mathscr{A} \to \mathbb{R}^m$ is a *measure* if $\mu(\emptyset) = 0$ and for any sequence $\{A_h\}_{h \in \mathbb{N}}$ of pairwise disjoint elements of $\mathscr{A}$,
$$\mu\left(\bigcup_{h=0}^{\infty} A_h\right) = \sum_{h=0}^{\infty} \mu(A_h).$$
If $m = 1$ we say that μ is a *real measure*; if $m > 1$ we say that μ is a *vector measure*;
- if μ is a measure, we define its *total variation* $|\mu|$ for every $A \in \mathscr{A}$ as
$$|\mu|(A) = \sup\left\{\sum_{h=0}^{\infty} |\mu(A_h)| : A_h \in \mathscr{A} \text{ pairwise disjoint}, A = \bigcup_{h=0}^{\infty} A_h\right\};$$
- if μ is a real measure, we define its *positive* and *negative parts* respectively as follows:
$$\mu^+ = \frac{|\mu| + \mu}{2} \quad \text{and} \quad \mu^- = \frac{|\mu| - \mu}{2}.$$
which are both positive finite measures: thus the Jordan decomposition for μ holds:
$$\mu = \mu^+ - \mu^-.$$

Theorem 1.4. *If μ is a measure on $(X, \mathscr{A})$, then $|\mu|$ is a positive finite measure.*

Remark 1.5. Positive measures differ from real measures since real measures must be finite.

Definition 1.6 (μ-negligible sets)**.** Let μ be a positive measure on the measure space $(X, \mathscr{A})$. Then

- a set $N \subset X$ is *μ-negligible* if there exists $A \in \mathscr{A}$ such that $N \subset A$ and $\mu(N) = 0$.
- a property $P(x)$ depending on the point $x \in X$ holds μ-a.e. in X if the set where P fails is a μ-negligible set;
- let $\mathscr{A}_\mu$ be the collection of all the subsets of X of the form $F = A \cup N$, with $A \in \mathscr{A}$ and N μ-negligible. Then $\mathscr{A}_\mu$ is a σ-algebra and $A \subset X$ is μ-measurable if $A \subset \mathscr{A}_\mu$. Moreover $\mu(F) = \mu(A)$.

Definition 1.7 (Measurable functions)**.** Let $(X,\mathscr{A})$ a measure space, (Y,d) a metric space. Let $u : X \to Y$ a function. Then

- u is *$\mathscr{A}$-measurable* on X if $u^{-1}(A) \in \mathscr{A}$ for every open set $A \subset Y$;
- if μ is a positive measure on $(X,\mathscr{A})$, u is *μ-measurable* if it is $\mathscr{A}_\mu$-measurable.

Definition 1.8. We say that $u : X \to \mathbb{R}$ is a *μ-simple function* if u is μ-measurable and the image of u is finite, i.e. if u belongs to the vector space generated by the characteristic functions.

Definition 1.9 (Integrals)**.** Let $(X,\mathscr{A})$ be a measure space.

- Let μ a positive measure on $(X,\mathscr{A})$ and $u : X \to [0,\infty]$ a simple μ-measurable function. Then the *integral* of u is defined by

  $$\int_X u \, \mathrm{d}\mu = \sum_{z \in u(X)} z \mu\left(u^{-1}(z)\right),$$

  with the convention that, whenever $z = 0$ and $\mu(u^{-1}(z)) = \infty$, the product $z\mu(u^{-1}(z)) = 0$. The definition is extended to any μ-measurable function $u : X \to [0,\infty]$ by setting:

  $$\int_X u \, \mathrm{d}\mu = \sup\left\{ \int_X v \, \mathrm{d}\mu : v \ \mu\text{-measurable, simple, } v \le u \right\}.$$

- A μ-measurable map $u : X \to \overline{\mathbb{R}}$ is *μ-summable* if

  $$\int_X |u| \, \mathrm{d}\mu < \infty.$$

  A μ-measurable map $u : X \to \overline{\mathbb{R}}$ is *μ-integrable* if either

  $$\int_X u^+ \, \mathrm{d}\mu < \infty \quad \text{or} \quad \int_X u^- \, \mathrm{d}\mu < \infty.$$

  If u is μ-integrable, we set

  $$\int_X u \, \mathrm{d}\mu = \int_X u^+ \, \mathrm{d}\mu - \int_X u^- \, \mathrm{d}\mu.$$

- Let μ be a measure on $(X,\mathscr{A})$ and $u : X \to \overline{\mathbb{R}}$ a $|\mu|$-measurable function; we say that u is *μ-summable* if u is *$|\mu|$-summable* and, if μ is real, we set

  $$\int_X u \, \mathrm{d}\mu = \int_X u \, \mathrm{d}\mu^+ - \int_X u \, \mathrm{d}\mu^-.$$

  If μ is a $\mathbb{R}^m$-valued vector measure then we set

  $$\int_X u \, \mathrm{d}\mu = \left(\int_X u \, \mathrm{d}\mu_1, \ldots, \int_X u \, \mathrm{d}\mu_m \right).$$

If μ is real and $u = (u_1, \ldots, u_k) : X \to \mathbb{R}^k$ is a $|\mu|$-measurable function, we say that u is $|\mu|$-*summable* if all its components are $|\mu|$-summable, and we set

$$\int_X u \, \mathrm{d}\mu = \left(\int_X u_1 \, \mathrm{d}\mu, \ldots, \int_X u_k \, \mathrm{d}\mu \right).$$

- When A is a μ-measurable set, the integral of a function u on A is defined by

$$\int_A u \, \mathrm{d}\mu = \int_X u \chi_A \, \mathrm{d}\mu.$$

Remark 1.10. As immediate consequence the following inequality holds for every extended real or vector valued summable function u and for every positive, real or vector measure μ:

$$\left| \int_X u \, \mathrm{d}\mu \right| \leq \int_X |u| \, \mathrm{d}|\mu|.$$

Definition 1.11 ($\mathrm{L^p}$ space)**.** Let $(X, \mathscr{A})$ be a measure space, μ a positive measure on it and $u : X \to \overline{\mathbb{R}}$ a μ-measurable function. We set

$$\|u\|_p = \begin{cases} \left(\int_X |u|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \inf\{C \in [0, \infty] : |u(x)| \leq C \text{ for } \mu\text{-a.e. } x \in X\} & \text{if } p = \infty. \end{cases}$$

We say that $u \in \mathrm{L}^p(X, \mu)$ if $\|u\|_p < \infty$. The set $\mathrm{L}^p(X, \mu)$ is a real vector space and $\|\cdot\|_p$ is a semi-norm.

Theorem 1.12 (Monotone convergence)**.** *Let $u_h : X \to \overline{\mathbb{R}}$ be an increasing sequence of μ-measurable functions, and assume that $u_h \geq g$, with $g \in \mathrm{L}^1(X, \mu)$, for any $h \in \mathbb{N}$. Then*

$$\lim_{h \to \infty} \int_X u_h \, \mathrm{d}\mu = \int_X \lim_{h \to \infty} u_h \, \mathrm{d}\mu.$$

Lemma 1.13 (Fatou)**.** *Let $u_h : X \to \overline{\mathbb{R}}$ be μ-measurable functions and $g \in \mathrm{L}^1(X, \mu)$. Then*

- *if $u_h \geq g$ for any $h \in \mathbb{N}$*

$$\int_X \liminf_{h \to \infty} u_h \, \mathrm{d}\mu \leq \liminf_{h \to \infty} \int_X u_h \, \mathrm{d}\mu;$$

- *if $u_h \leq g$ for any $h \in \mathbb{N}$*

$$\int_X \limsup_{h \to \infty} u_h \, \mathrm{d}\mu \geq \limsup_{h \to \infty} \int_X u_h \, \mathrm{d}\mu.$$

Theorem 1.14 (Dominated convergence)**.** *Let u, $u_h : X \to \overline{\mathbb{R}}$ be μ-measurable functions, and assume that $u_h(x) \to u(x)$ for μ-a.e. $x \in X$ as $h \to \infty$. If*

$$\int_X \sup_h |u_h| \, \mathrm{d}\mu < \infty$$

then

$$\lim_{h \to \infty} \int_X u_h \, \mathrm{d}\mu = \int_X u \, \mathrm{d}\mu.$$

Proposition 1.15. *Let μ be a positive measure on the measure space $(X, \mathscr{A})$ and let $u \in [\mathrm{L}^1(X,\mu)]^m$. We define the $\mathbb{R}^m$-valued measure*

$$u\,\mu(B) = \int_B u \,\mathrm{d}\mu, \quad \forall B \in \mathscr{A}.$$

Moreover, if $u\,\mu$ is the measure introduced above, then

$$|u\,\mu|(B) = \int_B |u| \,\mathrm{d}\mu, \quad \forall B \in \mathscr{A}.$$

Definition 1.16 (Absolute continuity)**.** Let μ be a positive measure and ν a real or vector measure on the measure space $(X, \mathscr{A})$. We say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if

$$\mu(B) = 0 \implies |\nu|(B) = 0, \quad \forall B \in \mathscr{A}.$$

Remark 1.17. If u is μ-summable, then the measure $u\,\mu$ is absolutely continuous with respect to μ.

Definition 1.18 (Singularity)**.** If μ, ν are positive measures, we say that they are *mutually singular*, and we write $\nu \perp \mu$, if there exists $A \in \mathscr{A}$ such that $\mu(A) = 0$ and $\nu(X \setminus A) = 0$; if μ or ν are real or vector valued, we say that they are *mutually singular* if $|\mu|$ and $|\nu|$ are so.

Definition 1.19 (Equi-integrability)**.** If $\mathscr{F} \subset \mathrm{L}^1(X,\mu)$ we say that $\mathscr{F}$ is an *equi-integrabile family* if the following two conditions hold:

- for any $\varepsilon > 0$ there exists a μ-measurable set A with $\mu(A) < \infty$ such that

$$\int_{X\setminus A} |u| \,\mathrm{d}\mu < \varepsilon, \qquad \text{for any } u \in \mathscr{F};$$

- for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for every μ-measurable set A, if $\mu(A) < \delta$ then

$$\int_A |u| \,\mathrm{d}\mu < \varepsilon, \quad \forall u \in \mathscr{F}.$$

Proposition 1.20. *Let $\mathscr{F} \subset \mathrm{L}^1(X,\mu)$. Then $\mathscr{F}$ is equi-integrable if and only if*

$$\{A_h\}_{h\in\mathbb{N}} \subset \mathscr{A}, \quad A_h \downarrow \emptyset \implies \lim_{h\to\infty} \sup_{u\in\mathscr{F}} \int_{A_h} |u| \,\mathrm{d}\mu = 0.$$

If μ is a finite measure and $\mathscr{F}$ is bounded in $\mathrm{L}^1(X,\mu)$, then $\mathscr{F}$ is equi-integrable if and only if

$$\mathscr{F} \subset \left\{ u \in \mathrm{L}^1(X,\mu) : \int_X \varphi(|u|) \,\mathrm{d}\mu \le 1 \right\}$$

for some increasing continuous function $\varphi : [0,\infty) \to [0,\infty]$ satisfying $\varphi/t \to \infty$ as $t \to \infty$ or equivalently if and only if

$$\lim_{t\to\infty} \sup_{u\in\mathscr{F}} \int_{\{|u|>t\}} |u| \,\mathrm{d}\mu = 0.$$

Theorem 1.21 (Radon-Nikodým)**.** *Let μ be a positive measure and ν a real or vector measure on the measure space $(X, \mathscr{A})$ and assume that ν is σ-finite. Then there is a unique pair of $\mathbb{R}^m$-valued measures ν^a, ν^s such that $\nu^a \ll \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$. Moreover, there is a unique function $u \in [\mathrm{L}^1(X, \mu)]^m$, such that $\nu^a = u\,\mu$. The function u is called the density of ν with respect to μ and is denoted by ν/μ.*

Theorem 1.22 (Fubini)**.** *Let $(X_1, \mathscr{A}_1)$, $(X_2, \mathscr{A}_2)$ be measure spaces and μ_1, μ_2 be positive σ-finite measures in X_1, X_2 respectively. Then there is a unique positive σ-finite measure μ on $(X_1 \times X_2, \mathscr{A}_1 \times \mathscr{A}_2)$ such that*

$$\mu(A_1 \times A_2) = \mu(A_1) \cdot \mu(A_2), \quad \forall A_1 \in \mathscr{A}_1, \forall A_2 \in \mathscr{A}_2.$$

Furthermore, for any μ-measurable function $u : X_1 \times X_2 \to [0, \infty]$ we have that

$$x \longmapsto \int_{X_2} u(x,y)\,\mathrm{d}\mu_2(y) \quad \textit{and} \quad y \longmapsto \int_{X_1} u(x,y)\,\mathrm{d}\mu_1(x)$$

are respectively μ_1-measurable and μ_2-measurable and

$$\int_{X_1 \times X_2} u\,\mathrm{d}\mu = \int_{X_1} \left(\int_{X_2} u(x,y)\,\mathrm{d}\mu_2(y) \right) \mathrm{d}\mu_1(x) = \int_{X_2} \left(\int_{X_1} u(x,y)\,\mathrm{d}\mu_1(x) \right) \mathrm{d}\mu_2(y).$$

1.2 Borel and Radon measure

Definition 1.23 (Borel σ-algebra)**.** The *Borel σ-algebra* of X is the smallest σ-algebra of X containing the open subsets of X.

Definition 1.24. Let X be a locally compact and separable metric space, $\mathscr{B}$ its Borel σ-algebra and consider the measure space $(X, \mathscr{B}(X))$. Then

- a *Borel measure* is a positive measure on $(X, \mathscr{B}(X))$, i.e. every Borel set is μ-measurable;
- a Borel measure μ is *regular* if for every $A \subset X$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$. Thus, a Borel measure is completely determined by its values on Borel sets;
- a *positive Radon measure* is a Borel measure *locally finite*, i.e. $\mu(K) < \infty$ with K compact sets;
- a (real or vector) set function defined on the relatively compact Borel subsets of X, that is a measure on $(K, \mathscr{B}(K))$ for every compact set $K \subset X$ is called a (*real* or *vector*) *Radon measure* on X. If $\mu : \mathscr{B}(X) \to \mathbb{R}^n$ is a measure, we say that is a *finite Radon measure*.

Example 1.25. The Lebesgue measure on X and the Dirac measure δ_x at $x \in X$ are well-known examples of Radon measure on $\mathbb{R}^n$.

Definition 1.26 (Borel functions)**.** Let X, Y be metric spaces, and let $u : X \to Y$. We say that u is a *Borel function* if $u^{-1}(A) \in \mathscr{B}(X)$ for every open set $A \subset Y$.

Theorem 1.27 (Inner and outer regularity of measures)**.** *Let X a locally compact and separable metric space and μ a Borel measure on X. Let A a μ-measurable set. Then*

- *if μ is σ-finite, then $\mu(A) = \sup\{\mu(K) : K \subset A, K$ compact$\}$;*
- *assume that a sequence $\{X_h\}_{h\in\mathbb{N}}$ of open sets in X exists such that $\mu(X_h) < \infty$ for any h and $X = \bigcup_{h\in\mathbb{N}} X_h$; then $\mu(A) = \inf\{\mu(E) : E \subset A, A$ open$\}$.*

Theorem 1.28 (Lusin)**.** *Let X be a locally compact and separable metric space and μ a Borel measure on X. Let $u : X \to \mathbb{R}$ be a μ-measurable function vanishing outside of a set with finite measure. Then, for any $\varepsilon > 0$ there exists a continuous function $v : X \to \mathbb{R}$ such that*

- *$||v||_\infty < ||u||_\infty$;*
- *$\mu(\{x \in X : v(x) \neq u(x)\}) < \varepsilon$.*

Remark 1.29. If μ is a finite Borel measure on X, an easy consequence is that for any μ-measurable function $u : X \to \mathbb{R}$ there exists a sequence $\{K_h\}_{h\in\mathbb{N}}$ of compact sets in X such that

$$\mu\left(X \setminus \bigcup_{h=0}^{\infty} K_h\right) = 0,$$

and $u|_{K_h}$ is continuous for every h.

Proposition 1.30. *Let X be a locally compact and separable metric space and μ a finite $\mathbb{R}^m$-valued Radon measure on it. Then for every open set $A \subset X$ the following equality holds:*

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{m}\int_X u_i \, \mathrm{d}\mu_i : u \in [C_c(A)]^m, ||u||_\infty \leq 1\right\}.$$

1.3 Outer measure

Definition 1.31 (Outer measure)**.** Let X a metric space and $\mathscr{P}(X)$ the collection of all the subset of X. A mapping $\mu : \mathscr{P}(X) \to [0, \infty]$ is called an *outer measure* on X if $\mu(\emptyset) = 0$, μ is σ-subadditivity and

$$\operatorname{dist}(A, F) > 0 \implies \mu(A \cup F) = \mu(A) + \mu(F), \quad \text{for any } A, F \subset X.$$

Remark 1.32. The *Lebesgue measure* of a set $A \in \mathbb{R}^n$ is an outer measure and it is definied as

$$\mathscr{L}^n(A) = \inf_{\mathscr{A}} \sum_{\mathscr{Q}\in\mathscr{A}} r(\mathscr{Q})^n,$$

where $\mathscr{A}$ is a countable covering of A by cubes with sides parallel to the coordinate axis, and $r(\mathscr{Q})$ denotes the side length of $\mathscr{Q}$. This measure is usually interpreted as the n-dimensional volume of A and we write $\mathscr{L}^n(A) = |A|$, with $|A| =$ *volume* of A.

Remark 1.33. The Hausdorff measure (Section 1.6) is an outer measure.

Definition 1.34. Let μ be an outer measure on X and $A \subset X$. Then μ *restricted* to A, written $\mu \llcorner A$, is the measure defined by

$$(\mu \llcorner A)(B) = \mu(A \cap B), \quad \forall B \subset X.$$

Remark 1.35. If A is a Borel set, then $\mu \llcorner A$ is a Borel regular measure, even if $\mu(A) = \infty$.

Theorem 1.36. *Let μ be a Borel regular measure on $\mathbb{R}^n$. Suppose $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$. Then $\mu \llcorner A$ is a Radon measure. So we can generate a Radon measure by restricting μ to a measurable set of finite measure.*

Theorem 1.37 (Carathéodory criterion)**.** *Let μ be an outer measure on the metric space X; then μ is σ-additive on $\mathscr{B}(X)$, hence the restriction of μ to Borel sets of X is a positive measure.*

Remark 1.38. The Carathéodory criterion shows that an outer measure always defines a Borel measure, so a Lebesgue measure is a Borel measure; conversely, if μ is a Borel measure on X, then it can be extended to every $A \subset X$ by setting

$$\mu(A) = \inf\{\mu(B) : B \in \mathscr{B}(X), B \supset A\}.$$

Moreover, if μ is an outer measure on X and $A \subset B \subset X$, then $\mu(A) \leq \mu(B)$.

Theorem 1.39 (Carathéodory's Theorem)**.** *If μ is an outer measure on X and $\mathscr{M}(\mu)$ is the family for those $A \subset X$ such that*

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A), \quad \forall B \subset X,$$

then $\mathscr{M}(\mu)$ is a σ-algebra, and μ a measure on $\mathscr{M}(\mu)$. Elements of $\mathscr{M}(\mu)$ are called μ-measurable sets.

Theorem 1.39 states that every outer measure on X can be seen as a measure on a σ-algebra on X. In this way, various classical results from Measure Theory are immediately recovered in the context of outer measure.

1.4 Riesz Theorem and weak-$*$ convergence in measure

Definition 1.40 (Support)**.** Let μ be a positive measure on the locally compact and separable metric space X. We call the closed set of all points $x \in X$, such that $\mu(A) > 0$, for any neighbourhood A of x, the *support of* μ and we denote it by $\operatorname{supp} \mu$. If μ is a real or vector measure, we call the support of μ, the support of $|\mu|$.

In the following, we will denote by $C_c^0(X)$ the space of continuous functions with compact support and by $C_0(X)$ its completion with respect to the sup-norm.

Definition 1.41 (Total Variation)**.** The *total variation* of a linear functional L on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$ is the set function $|L|:\mathscr{P}(\mathbb{R}^n)\to[0,\infty]$ such that, for $A\subset\mathbb{R}^n$ open,

$$|L|(A)=\sup\Big\{\langle L,\varphi\rangle:\varphi\in C_c^0(A;\mathbb{R}^m),\,|\varphi|\le 1\Big\},$$

and, for $E\subset\mathbb{R}^n$ arbitrary,

$$|L|(E)=\inf\Big\{|L|(A):E\subset A\text{ and }A\text{ is open}\Big\}.$$

Theorem 1.42 (Riesz)**.** *Let X be a locally compact and separable metric space; suppose that the functional $L:[C_0(X)]^m\to\mathbb{R}$ is additive and bounded, i.e. satisfies the following conditions:*

- $L(u+v)=L(u)+L(v)$ *for any* $u,\,v\in[C_0(X)]^m$*;*
- $||L||=\sup\{L(u):u\in[C_0(X)]^m,\,|u|\le 1\}<\infty.$

Then, there is a unique $\mathbb{R}^m$-valued finite Radon measure μ on X such that

$$L(u)=\sum_{h=1}^m\int_X u_h\,\mathrm{d}\mu_h,\quad\forall u\in[C_0(X)]^m,$$

and $||L||=|\mu|(X)$, a Radon measure.

Example 1.43 (Riemann integral)**.** Let $X=\mathbb{R}$ and $L:C_c^0(\mathbb{R})\to\mathbb{R}$ the linear functional which assigns to $u\in C_c^0(\mathbb{R})$ its Riemann integral; according to Riesz Theorem, L defines a measure on $\mathbb{R}$. This is a way to construct the Lebesgue measure.

Remark 1.44. From the Riesz Theorem, the space of Radon measure on $\mathbb{R}^n$ is identified with the dual space of $C_c^0(\mathbb{R}^n)$. So, the Riesz Theorem can be restated by saying that the dual of the Banach space $[C_0(X)]^m$ is the space $[\mathscr{M}(X)]^m$ of finite $\mathbb{R}^m$-valued Radon measures on X, under the pairing

$$(u,\mu)=\sum_{h=1}^m\int_X u_h\,\mathrm{d}\mu_h.$$

Proposition 1.45 (Convergence in measure)**.** *Let u_h and u be μ-measurable functions. We say that $\{u_h\}_{h\in\mathbb{N}}$ converges to u in measure if*

$$\lim_{h\to\infty}\mu(\{x\in X:|u_h(x)-u(x)|>\varepsilon\})=0,\quad\forall\varepsilon>0.$$

Definition 1.46 (Weak-$*$ convergence)**.** Let $\mu\in[\mathscr{M}_{\mathrm{loc}}(X)]^m$ and let $\{\mu_h\}_{h\in\mathbb{N}}\subset[\mathscr{M}_{\mathrm{loc}}(X)]^m$; we say that $\{\mu_h\}_{h\in\mathbb{N}}$ *locally weak-$*$ converges* to μ if

$$\lim_{h\to\infty}\int_X u\,\mathrm{d}\mu_h=\int_X u\,\mathrm{d}\mu,\quad\text{for every }u\in C_c(X);$$

if μ and μ_h are finite, we say that $\{\mu_h\}_{h\in\mathbb{N}}$ *weak-$*$ converges* to μ if

$$\lim_{h\to\infty}\int_X u\,\mathrm{d}\mu_h=\int_X u\,\mathrm{d}\mu,\quad\text{for every }u\in C_0(X).$$

Theorem 1.47 (Weak-$*$ compactness). *If $\{\mu_h\}_{h\in\mathbb{N}}$ is a sequence of finite Radon measures on the locally compact and separable metric space X with $\sup\{|\mu_h|(X) : h \in \mathbb{N}\} < \infty$, then it has a weakly-$*$ converging subsequence and the map $\mu \to |\mu|(X)$ is lower semicontinuous with respect to the weak-$*$ convergence.*

Proposition 1.48. *Let $\{\mu_h\}_{h\in\mathbb{N}}$ be a sequence of Radon measures on the locally compact and separable metric space X locally weakly-$*$ converging to μ. Then*

- *if the measures μ_h are positive, then for every lower semicontinuous function $u : X \to [0,\infty]$*
$$\liminf_{h\to\infty} \int_X u \,\mathrm{d}\mu_h \geq \int_X u \,\mathrm{d}\mu$$
and for every upper semicontinuous function $v : X \to [0,\infty)$ with compact support
$$\limsup_{h\to\infty} \int_X v \,\mathrm{d}\mu_h \leq \int_X v \,\mathrm{d}\mu;$$
- *if $|\mu_h|$ locally weakly-$*$ converges to λ, then $\lambda \geq |\mu|$. Moreover, if A is a relatively compact μ-measurable set such that $\lambda(\partial A) = 0$, then $\mu_h(A) \to \mu(A)$ as $h \to \infty$. More generally*
$$\int_X u \,\mathrm{d}\mu = \lim_{h\to\infty} \int_X u \,\mathrm{d}\mu_h$$
for any bounded Borel function $u : X \to \mathbb{R}$ with compact support such that the set of its discontinuity points is λ-negligible.

Example 1.49. Considering the characteristic function of compact and open sets. From Proposition 1.48 we have, whenever $\{\mu_h\}_{h\in\mathbb{N}}$ is locally weak-$*$ convergent to μ,

- $\mu(K) \geq \limsup_h \mu_h(K)$ if K compact;
- $\mu(A) \leq \liminf_h \mu_h(A)$ if A open.

Proposition 1.50. *Let $(X, \mathscr{A})$ be a measure space, μ a positive measure on it and $u : X \to [0,\infty]$ a μ-measurable function. Then*
$$\int_X u \,\mathrm{d}\mu = \int_0^\infty \mu\{x \in X : u(x) > t\} \,\mathrm{d}t.$$

Proof. If $\{u > 0\}$ is not σ-finite with respect to μ, then both sides are ∞; otherwise, possibly replacing μ by $\mu \llcorner \{u > 0\}$, we can assume that μ is a σ-finite measure. We apply Fubini's Theorem in $X \times (0,\infty)$ with $\mu_1 = \mu$ and $\mu_2 = \mathscr{L}^1$, the Lebesgue measure on real line. Let
$$E_t = \{x \in X : u(x) > t\},$$
then we have
$$\int_X u \,\mathrm{d}\mu = \int_X \left(\int_0^\infty \chi_{E_t}(x) \,\mathrm{d}t \right) \mathrm{d}\mu(x) = \int_0^\infty \left(\int_X \chi_{E_t}(x) \,\mathrm{d}\mu(x) \right) \mathrm{d}t = \int_0^\infty \mu(E_t) \,\mathrm{d}t. \qquad \square$$

Proposition 1.51. *Let $\{\mu_h\}_{h\in\mathbb{N}}$ be a sequence of positive Radon measures on X, and assume the existence of a positive, finite Radon measure μ in X such that*

$$\lim_{h\to\infty}\mu_h(X)=\mu(X) \quad \textit{and} \quad \liminf_{h\to\infty}\mu_h(A)\geq\mu(A)$$

for every $A\subset X$ open. Then

$$\lim_{h\to\infty}\int_X u\,\mathrm{d}\mu_h=\int_X u\,\mathrm{d}\mu$$

for any bounded continuous function $u:X\to\mathbb{R}$. In particular $\{\mu_h\}_{h\in\mathbb{N}}$ weakly-$*$ converges to μ in X.

1.5 Differentiation of a measure

Definition 1.52. For positive Radon measures μ,ν and $x\in\operatorname{supp}\mu$ we define the following Borel functions:

$$D^+_\mu\nu(x)=\limsup_{\rho\to0}\frac{\nu(B(x,\rho))}{\mu(B(x,\rho))} \quad\text{and}\quad D^-_\mu\nu(x)=\liminf_{\rho\to0}\frac{\nu(B(x,\rho))}{\mu(B(x,\rho))}.$$

Remark 1.53. Since open balls can be approximated from inside by closed balls and closed balls can be approximated from outside by open balls, the densities $D^\pm_\mu$ do not change if we replace open balls by closed balls.

Proposition 1.54. *Let μ and ν be positive Radon measures in $\mathbb{R}^n$ and let $t\in[0,\infty)$. For any Borel set $A\subset\operatorname{supp}\mu$ the following two implications hold:*

$$D^-_\mu\nu(x)\leq t,\ \forall x\in A \implies \nu(A)\leq t\,\mu(A),$$
$$D^+_\mu\nu(x)\geq t,\ \forall x\in A \implies \nu(A)\geq t\,\mu(A).$$

In particular, if ν is finite then $\mu(\{x:D^+_\mu(x)=\infty\})=0$.

Theorem 1.55 (Bezicovič derivation theorem)**.** *Let μ a positive Radon measure in an open set $\Omega\subset\mathbb{R}^n$, and ν an $\mathbb{R}^m$-valued Radon measure. Then, for μ-a.e. x in the support of μ the limit*

$$u(x)=\lim_{\rho\to0}\frac{\nu(B(x,\rho))}{\mu(B(x,\rho))}$$

exists in $\mathbb{R}^m$ and moreover the Radon-Nykodým decomposition of ν is given by $\nu=u\mu+\nu^s$, where $\nu^s=\nu\llcorner A$ and A is the μ-negligible set

$$A=(\Omega\setminus\operatorname{supp}\mu)\cup\left\{x\in\operatorname{supp}\mu:\lim_{\rho\to0}\frac{|\nu|(B(x,\rho))}{\mu(B(x,\rho))}=\infty\right\}.$$

Remark 1.56. Theorem 1.55 gives a concrete representation of the density ν/μ.

Corollary 1.57 (Lebesgue points)**.** *Let μ be a positive Radon measure in an open set $\Omega\subset\mathbb{R}^n$ and $u\in\mathrm{L}^1(\Omega,\mu)$. Then for μ-a.e. $x\in\Omega$ the following equality holds:*

$$\lim_{\rho\to0^+}\frac{1}{\mu(B(x,\rho))}\int_{B(x,\rho)}|u(y)-u(x)|\,\mathrm{d}\mu=0.$$

In this case, we say that x is a Lebesgue point of u with respect to μ.

1.6 Hausdorff Measure

The Hausdorff dimension was firstly introduced in 1918 by Felix Hausdorff, and then exploited by Abram Bezicovič: that's why it is also called the *Hausdorff-Besicovič* dimension. Roughly speaking, the dimension of a set is the number of indipendent parameters necessary to describe a point in the set. For example, a point in a plane is described by two Cartesian Coordinates (so the plane is bidimensional). We can construct strange sets in a plane for which the dimension is lower: the most famous set is the Sierpinski triangle whose dimension is $\ln(3)/\ln(2)$. The Hausdorff dimension is the right tool to calculate this dimension, which is a real number and it is strictly connected with *fractal dimension*, investigated by Benoît Mandelbrot.

Definition 1.58 (Fractal dimension)**.** Given an unitary object with linear dimension equal to 1 in the Euclidean dimension D, we can reduce the linear dimension by a factor $1/l$ in every spatial direction obtaining $N = l^D$ similar objects which can be used to rebuild the original object. So the fractal dimension is defined by

$$D = \frac{\log N(l)}{\log l}.$$

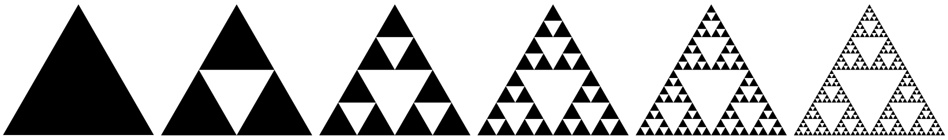


Figure 1: The Sierpinski triangle formation process. In this case a line in linear dimension equal to 1 is reduced by a factor 1/2 in every direction. So $l = 2$ and we obtain $N = 3$ similar object.

From now on we restrict our consideration to $\mathbb{R}^n$ remembering that by the *Whitney extension theorem* a lot of cases can be reconducted to this one: if A is a closed subset of an Euclidean space, then it is possible to extend a given function off A in such a way as to have prescribed derivatives at the points of A.

Definition 1.59. (Hausdorff measure) Let $k \in [0, +\infty)$ and $A \subset \mathbb{R}^n$. The k-dimensional Hausdorff measure of A is given by

$$\mathscr{H}^k(A) = \lim_{\delta \to 0^+} \mathscr{H}^k_\delta(A)$$

where, for $0 < \delta < \infty$, $\mathscr{H}^k_\delta(A)$ is the outer measure defined by

$$\mathscr{H}^k_\delta(A) = \frac{\omega_k}{2^k} \inf \left\{ \sum_{i \in I} [\mathrm{diam}(A_i)]^k : \mathrm{diam}(A_i) < \delta, A \subset \bigcup_{i \in I} A_i \right\}$$

for finite or countable covers $\{A_i\}_{i \in I}$, with $\frac{\omega_k}{2^k}$ as normalization factor and the convention $\mathrm{diam}(\emptyset) = 0$.

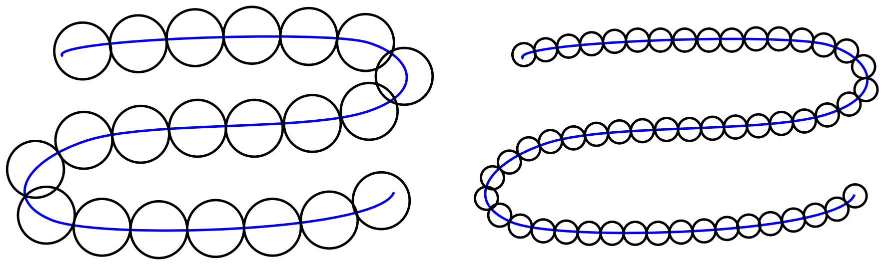


Figure 2: Graphic example of $\mathcal{H}^k_\delta(A)$, where A is the blue curve. Covering with smaller sets is needed to compute the length where the curvature is higher: this follows the local geometry of A.

Remark 1.60. We define ω_k as

$$\omega_k = \frac{\pi^{\frac{k}{2}}}{\Gamma\left(1+\frac{k}{2}\right)}, \quad \text{where } \Gamma(t) = \int_0^\infty s^{t-1} e^{-s}\, ds \text{ is the Euler } \Gamma \text{ function.}$$

Definition 1.61 (Curve)**.** A set $\Gamma \subset \mathbb{R}^n$ is a curve if there exist $a > 0$ and a continuous injective function $\gamma : [0,a] \to \mathbb{R}^n$ such that $\Gamma = \gamma([0,a])$. In this case, γ is called a *parametrization* of Γ. Given a parametrization $\gamma : [0,a] \to \mathbb{R}^n$ and a subinterval $[b,c]$ of $[0,a]$, we define the *length* of γ over $[b,c]$ as

$$\ell(\gamma;[b,c]) = \sup\left\{ \sum_{h=1}^{n} |\gamma(t_h) - \gamma(t_{h-1})| : b = t_0 < t_{h-1} < t_h < t_n = c, n \in \mathbb{N} \right\}.$$

So the length of Γ is $\text{length}(\Gamma) = \ell(\gamma;[0,a])$ and it is independent of the parametrization γ of Γ.

Definition 1.62 (Hausdorff dimension)**.** The Hausdorff dimension of $A \in \mathbb{R}^n$ is given by:

$$\mathcal{H}_{\dim}(A) = \inf\left\{ k \geq 0 : \mathcal{H}^k(A) = 0 \right\}.$$

The notion of dimension is justified by the following statements:

- if $A \subset \mathbb{R}^n$, then $\mathcal{H}_{\dim}(A) = [0,n]$. Moreover $\mathcal{H}^k(A) = \infty$ for every $k < \mathcal{H}_{\dim}(A)$;
- $\mathcal{H}^0$ is the counting measure;
- if A is a curve, then $\mathcal{H}^1(A)$ coincides with the classical length of A;
- if $1 \leq k \leq n-1$, $k \in \mathbb{N}$, and A is a k-dimensional C^1 surface, then $\mathcal{H}^k(A)$ coincides with the classical k-dimensional area of A;
- if $A \in \mathbb{R}^n$, then $\mathcal{H}^n(A) = \mathcal{L}^n(A)$;
- if $k > n$, then $\mathcal{H}^k = 0$;
- if A is an open set in $\mathbb{R}^n$, then $\mathcal{H}_{\dim}(A) = n$;
- for every $k \in [0,n]$ there exists a compact set K such that $\mathcal{H}_{\dim}(K) = k$.

Theorem 1.63 (Isodiametric inequality). *Among all sets of fixed diameter, balls have maximum volume. In other words,*

$$\mathscr{L}^n(A) \le \omega_n \left(\frac{\operatorname{diam}(A)}{2} \right)^n,$$

for any $\mathscr{L}^n$-measurable set $A \subset \mathbb{R}^n$.

Theorem 1.64. *The measures $\mathscr{L}^n$, $\mathscr{H}^n$ and $\mathscr{H}^n_\delta$ coincides in $\mathbb{R}^n$ so, for any Borel set $A \subset \mathbb{R}^n$ and an $\delta \in (0, \infty]$, there holds*

$$\mathscr{L}^n(A) = \mathscr{H}^n_\delta(A) = \mathscr{H}^n(A).$$

1.7 Topology convergence

Definition 1.65 (Strong convergence). A sequence $\{x_h\}_{h\in\mathbb{N}}$ in A converges strongly to $x \in A$, and we write $x_h \to x$, if

$$||x_h - x|| \to 0.$$

Definition 1.66 (Weak convergence). The weak topology $\sigma(A, A^*)$ on A is the coarsest topology on A (i.e. with the minimum number of open sets) such that all the elements in A^* are continuous. If a sequence $\{x_h\}_{h\in\mathbb{N}}$ in A converges to $x \in A$ in the weak topology $\sigma(A, A^*)$ we shall write $x_h \rightharpoonup x$, i.e.

$$\langle \varphi, x_h \rangle_{A^*,A} \to \langle \varphi, x \rangle \text{ in } \mathbb{R}, \quad \forall \varphi \in A^*.$$

Proposition 1.67. *When A is finite-dimensional, the weak topology $\sigma(A, A^*)$ and the usual topology are the same. In particular, a sequence $\{x_h\}_{h\in\mathbb{N}}$ converges weakly if and only if it converges strongly.*

So far, we have two topologies on A^*:

- the usual (strong) topology associated to the norm of A^*,
- the weak topology $\sigma(A^*, A^{**})$.

We are now going to define a third topology on A^* called the weak-$*$ topology and denoted by $\sigma(A^*, A)$ (the $*$ is to reminds us that this topology is defined only on dual spaces).

Definition 1.68 (Weak-$*$ convergence). The weak-$*$ topology $\sigma(A^*,A)$ on A^* is the smallest topology in which all the function of the set $J_A = \{Jx : x \in A\}$ are continuous. Since $J_A \subseteq A^{**}$, such topology is weaker than the weak topology $\sigma(A^*,A^{**})$ in which all the functions of A^{**} are continuous. Let $\{\varphi_h\}_{h\in\mathbb{N}}$ a sequence in A^*, $\varphi \in A^*$ and $x \in A$, then $\{\varphi_h\}_{h\in\mathbb{N}}$ converges to φ in the weak-$*$ topology, and we write $\varphi_h \overset{*}{\rightharpoonup} \varphi$, if and only if

$$\langle \varphi_h, x \rangle_{A^*,A} \to \langle \varphi, x \rangle_{A^*,A}, \quad \text{for all } x \in A.$$

Since $A \subset A^{**}$, it is clear that the topology $\sigma(A^*, A)$ has fewer open sets than the topology $\sigma(A^*, A^{**})$, which in turn has fewer open sets than the strong topology. Obviously, a coarser topology has more compact sets.

Proposition 1.69. *When A is finite-dimensional, the weak-topology $\sigma(A, A^*)$ and the usual topology are the same. In particular, a sequence $\{x_h\}_{h\in\mathbb{N}}$ converges weakly if and only if it converges strongly.*

Theorem 1.70. *Let C be a convex subset of A. Then, C is closed in the weak topology $\sigma(A, A^*)$ if and only if it is closed in the strong topology.*

Theorem 1.71 (Mazur)**.** *Assume $\{x_h\}_{h\in\mathbb{N}}$ converges weakly to x. Then there exists a sequence $\{y_h\}_{h\in\mathbb{N}}$ made up of convex combinations of the x_h's that converges strongly to x.*

Theorem 1.72. *Assume that $u : A \to (-\infty, +\infty)$ is convex and l.s.c. in the strong topology. Then φ is l.s.c. in the weak topology $\sigma(A, A^*)$.*

Proposition 1.73. *The weak-$*$ topology is Hausdorff.*

Ennio De Giorgi introduced Γ-convergence to minimize the functional $F(u)$, whenever it is singular and also its approximates are difficult to minimize.

Definition 1.74. Let X be a topological space and $\{F_h\}_{h\in\mathbb{N}} : X \to [0, +\infty)$ a sequence of functionals on X. Then $\{F_h\}_{h\in\mathbb{N}}$ is said to Γ-converge to the Γ-limit $F : X \to [0, +\infty)$ if the following two conditions hold:

- *Lower bound inequality*: For every sequence $\{x_h\}_{h\in\mathbb{N}} \in X$ such that $x_h \to x$ as $h \to \infty$,
$$F(x) \le \liminf_{h\to\infty} F_h(x_h).$$
- *Upper bound inequality*: For every $x \in X$, there is a sequence $\{x_h\}_{h\in\mathbb{N}}$ converging to x such that
$$F(x) \ge \limsup_{h\to\infty} F_h(x_h)$$

The first condition means that F provides an asymptotic common lower bound for the F_h. The second condition means that this lower bound is optimal.

Property 1.75. The following properties holds:

- Minimizers converge to minimizers: If $\{F_h\}_{h\in\mathbb{N}}$ Γ-converge to F, and x_h is a minimizer for $\{F_h\}_{h\in\mathbb{N}}$, then every cluster point of the sequence $\{x_h\}_{h\in\mathbb{N}}$ is a minimizer of F.
- Γ-limits are always lower semicontinuous.
- Γ-convergence is stable under continuous perturbations: If $\{F_h\}_{h\in\mathbb{N}}$ Γ-converges to F and $G : X \to [0, +\infty)$ is continuous, then $F_h + G$ will Γ-converge to $F + G$.
- A constant sequence of functionals $F_h = F$ does not necessarily Γ-converge to F, but to the relaxation of F, the largest lower semicontinuous functional below F.

1.8 Sobolev spaces

Definition 1.76 (Weak derivatives)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $i \in \{1,\dots,N\}$, $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$; if there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = -\int_\Omega g\varphi \,\mathrm{d}x, \quad \forall \varphi \in \mathrm{C}_c^\infty(\Omega)$$

then we say that u has weak i-th derivative given by g. This is unique and denoted by $\nabla_i u$ or $\partial u / \partial x_i$.

Remark 1.77. The weak derivatives coincide with the classical ones if $u \in C^1(\Omega)$.

Definition 1.78 (Sobolev spaces)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set, and $1 \le p \le \infty$; we say that $u \in \mathrm{W}^{1,p}(\Omega)$ if $u \in \mathrm{L}^p(\Omega)$ and has weak derivatives in $\mathrm{L}^p(\Omega)$ for every $i = 1,\dots,n$. For any $u \in \mathrm{W}^{1,p}(\Omega)$ we set

$$\nabla u = (\nabla_1 u, \dots, \nabla_n u).$$

We recall that $\mathrm{W}^{1,p}(\Omega)$ becomes a Banach space (Hilbert for $p = 2$) when endowed with the norm $||\cdot||_{\mathrm{W}^{1,p}(\Omega)}$ defined by

$$||u||_{\mathrm{W}^{1,p}(\Omega)} = \begin{cases} \left(||u||_p^p + \sum_{i=1}^n ||\nabla_i u||_p^p \right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty; \\ ||u||_\infty + \sum_{i=1}^n ||\nabla_i u||_\infty & \text{if } p = \infty. \end{cases}$$

Remark 1.79. The space $\mathrm{W}^{1,p}(\Omega)$ is separable for $1 \le p < \infty$ and reflexive for $1 < p < \infty$.

Proposition 1.80. *Let $\Omega \subset \mathbb{R}^n$ be open, and $\{u_h\}_{h\in\mathbb{N}}$ a sequence in $\mathrm{W}^{1,p}(\Omega)$ converging in $\mathrm{L}^p(\Omega)$ to some function u; then the following statements hold:*

- *if $1 \le p \le \infty$, and for every $i \in \{1,\dots,n\}$ there is $g_i \in \mathrm{L}^p(\Omega)$ such that $\nabla_i u_h \to g_i$ in $\mathrm{L}^p(\Omega)$, then $u \in \mathrm{W}^{1,p}(\Omega)$ and $g_i = \nabla_i u$;*

- *if $1 < p \le \infty$ and the sequences $\{\nabla_i u_h\}_{h\in\mathbb{N}}$ are bounded for any $i = 1,\dots,n$, then $u \in \mathrm{W}^{1,p}(\Omega)$ and $\nabla_i u_h \to \nabla_i u$ weakly (weakly-$*$ if $p = \infty$) for any $i = 1,\dots,n$.*

Definition 1.81. We denote by $\mathrm{W}_0^{1,\mathrm{p}}(\Omega)$ the closure of $\mathrm{C}_c^\infty(\Omega)$ in $\mathrm{W}^{1,p}(\Omega)$.

Theorem 1.82. *Let $I \subset \mathbb{R}$ be a bounded interval, $1 \le p \le \infty$, and let $u \in \mathrm{W}^{1,p}(I)$; then there is a unique function $\widetilde{u} \in \mathrm{C}(\overline{I})$ such that $\widetilde{u}(x) = u(x)$ for $\mathscr{L}^1$-a.e. $x \in I$ and*

$$\widetilde{u}(b) - \widetilde{u}(a) = \int_a^b u'(x)\,\mathrm{d}x, \quad \forall a, b \in I.$$

Definition 1.83 (Weak convergence in $W^{1,p}$)**.** Let $\Omega \subset \mathbb{R}^n$, $1 \le p \le \infty$ and u, $u_h \in W^{1,p}(\Omega)$; then, we say that $u_h \to u$ weakly in $W^{1,p}(\Omega)$ (weakly-$*$ if $p = \infty$) if ∇u_h weakly converge in $L^p(\Omega)$ (weakly-$*$ if $p = \infty$) and $u_h \to u$ strongly in $L^p(\Omega)$.

Definition 1.84 (Higher order Sobolev spaces)**.** Higher order weak derivatives $\nabla^\alpha u$ (with α multi-index) can be introduced, giving rise to the spaces $W^{k,p}$. If $u \in L^1_{\text{loc}}(\Omega)$ we say that $g \in L^1_{\text{loc}}(\Omega)$ is the α-th weak derivative of u if

$$\int_\Omega u \, \nabla^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega g \varphi \, dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Given an integer $k > 1$ and $1 \le p \le \infty$ the Sobolev space $W^{k,p}(\Omega)$ is thus defined as the set of functions $u \in L^p(\Omega)$ such that all weak derivatives $\nabla^\alpha u$ belong to $L^p(\Omega)$ for any $|\alpha| \le k$. It can be endowed with a norm,

$$||u||_{W^{k,p}(\Omega)} = \begin{cases} \left(||u||_p^p + \sum_{i=1}^n \sum_{|\alpha|=i} ||\nabla^\alpha u||_p^p \right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty; \\ ||u||_{W^{k,\infty}(\Omega)} = ||u||_\infty + \sum_{i=1}^n \sum_{|\alpha|=i} ||\nabla^\alpha u||_\infty & \text{if } p = \infty. \end{cases}$$

1.9 Lipschitz functions

We recall some basic results on Lipschitz functions, which are more flexible than C^1 functions, and provide useful properties arising from their sufficient degree of regularity.

Definition 1.85 (Lipschitz functions)**.** Let $A \subset \mathbb{R}^n$ and $u : A \to \mathbb{R}^n$. Then u is a Lipschitz function in A and we write $u \in [\text{Lip(A)}]^n$ if

$$\text{Lip(u, A)} \equiv \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|} : x \neq y \text{ and } x, y \in A \right\} < \infty.$$

By definition, Lip(u, A) is the least $L \in [0, \infty)$ such that

$$|u(x) - u(y)| \le L|x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

Proposition 1.86. *Let $A \subset \mathbb{R}^n$ and let $u : A \to \mathbb{R}$ be a Lipschitz function; then there is $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{u}(x) = u(x)$ for any $x \in A$ and* $\text{Lip}(\tilde{\text{u}}, \mathbb{R}^\text{n}) = \text{Lip(u, A)}$.

Remark 1.87. Since the Lipschitz property is preserved under mollification, any Lipschitz function $u : \Omega \to \mathbb{R}$ belongs to $W^{1,\infty}(\Omega)$ and satisfies $||\nabla u||_{L^\infty(\Omega)} \le \text{Lip(u, }\Omega)$. In general, however, $u \in W^{1,\infty}(\Omega)$ does not imply $u \in \text{Lip}(\Omega)$ because it may happen that $||\nabla u||_{L^\infty(\Omega)} < \text{Lip(u, }\Omega)$. The next proposition provides a sufficient condition ensuring the equality.

Proposition 1.88. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex, open set, and $u : \Omega \to \mathbb{R}$. Then $u \in \mathrm{W}^{1,\infty}(\Omega)$ if and only if* $\mathrm{Lip}(\mathrm{u}, \Omega) < \infty$ *and* $\|\nabla u\|_{\mathrm{L}^\infty(\Omega)} = \mathrm{Lip}(\mathrm{u}, \Omega)$.

Proposition 1.89 (Weak gradient of a Lipschitz function)**.** *If $u : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, then $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ and u admits a weak gradient $\nabla u \in \mathrm{L}^\infty(\mathbb{R}^n; \mathbb{R}^m \otimes \mathbb{R}^n)$.*

Theorem 1.90 (Rademacher's theorem)**.** *Any function $u \in \mathrm{W}^{1,\infty}(\Omega)$ is differentiable $\mathscr{L}^n$-a.e. in Ω and the differential coincides $\mathscr{L}^n$-a.e. with the weak derivative ∇u in Definition 1.85.*

Theorem 1.91. *Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $A \subset \mathbb{R}^n$, $0 \leq k < \infty$. Then*

$$\mathscr{H}^k(u(A)) \leq [\mathrm{Lip}(\mathrm{u})]^k \, \mathscr{H}^k(A).$$

In particular, $\mathscr{H}_{\dim}(u(A)) \leq \mathscr{H}_{\dim}(A)$, so the Hausdoorf measure decrease under projection over affine subspace of $\mathbb{R}^n$.

Lipschitz functions in Calculus of Variation

Let $I = [a, b] \subset \mathbb{R}$ an interval of $\mathbb{R}$ and $L : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a function. We denote the variables of L with (t, x, v) where $t \in I$, $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Let X a vectorial subspace of the space of functions defined on I with values in $\mathbb{R}^n$, a.e. differentiable. We consider the following problem, called *basic problem of Calculus of Variation*:

$$\inf_{x(\cdot) \in X} J(x), \quad J(x) = \int_I L(t, x(t), \dot{x}(t))\,\mathrm{d}t$$

Theorem 1.92 (Euler's equations)**.** *Given the basic problem of Calculus of Variation, with $X = C^2(I; \mathbb{R}^n)$ and $L \in C^2$, if $x(\cdot)$ is a solution, then it satisfy the Euler's equations:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial v_1}(t, x(t), \dot{x}(t)) = \dfrac{\partial L}{\partial x_1}(t, x(t), \dot{x}(t)), \\ \vdots \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial v_j}(t, x(t), \dot{x}(t)) = \dfrac{\partial L}{\partial x_j}(t, x(t), \dot{x}(t)), \quad j = 2, \ldots, n-1, \\ \vdots \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial v_n}(t, x(t), \dot{x}(t)) = \dfrac{\partial L}{\partial x_n}(t, x(t), \dot{x}(t)). \end{cases}$$

Definition 1.93 (Fréchet derivative)**.** Let X, Y normed spaces and Ω an open set in X. A function $u : \Omega \to Y$ is Fréchet-differentiable at $x_0 \in \Omega$ (or *F-differentiable at x_0*) if it exists a linear and continuous operator $A : X \to Y$, such that:

$$\lim_{\|h\|_X \to 0} \frac{u(x_0 + h) - u(x_0) - Ah}{\|h\|_X} = 0.$$

In this case, A is unique and is the *Fréchet derivative* of u at x_0, written as $A = u'(x_0) = Du(x_0)$.

Definition 1.94 (Gâteaux derivative)**.** Let X, Y normed spaces and Ω an open set in X. A function $u : \Omega \to Y$ is Gâteaux-differentiable at $x_0 \in \Omega$ (or *G-differentiable at* x_0) if it exists a linear and continuous operator $A : X \to Y$, such that:

$$\lim_{t \to 0} \frac{u(x_0 + h) - u(x_0)}{t} = Av.$$

In this case, A, if exists, is unique and it is the *Gâteaux differential* of u at x_0, written as $A = u'_G(x_0)$. The application $x \longmapsto u'_G(x)$ is the *Gâteaux derivative* of u.

Remark 1.95. There exist functions G-differential at x_0 which are not F-differential at x_0. However, if a function is F-differential at x_0, it is also G-differential at x_0 and $u'(x_0) = u'_G(x_0)$.

Definition 1.96 (Functions absolutely continuous)**.** A function u is *absolutely continuous* in $[a, b]$, written $u \in \mathrm{AC}([\mathrm{a}, \mathrm{b}])$ if there exist $v \in \mathrm{L}^1([a, b])$ such that

$$u(t) = u(a) + \int_a^t v(s)\,\mathrm{d}s, \quad \forall t \in [a, b].$$

The following relation holds:

$$\mathrm{W}^{1,1}([a, b]) = \mathrm{AC}([\mathrm{a}, \mathrm{b}]) \supset \mathrm{Lip}([\mathrm{a}, \mathrm{b}]).$$

Proposition 1.97. *Given the Euler's equations in integral form*

$$\nabla_v L(t, x(t), \dot{x}(t)) = c + \int_a^t \nabla_x L(s, x(s), \dot{x}(s))\,\mathrm{d}s,$$

if $(x, v) \longmapsto L(t, x, v)$ *is convex, then every weak extremal is a global minimum. If the map is strictly convex, the minimum is unique.*

Proposition 1.98. *Given the basic problem of Calculus of Variation on* $X = \mathrm{Lip}(\Omega)$*, if* $L(t, x, \cdot) \in C^1$ *is strictly convex, then the infimum is* $\overline{x} \in C^1$.

Theorem 1.99 (Hilbert-Weierstraß)**.** *Let* $L \in C^2$*,* $\partial^2_{v,v} L > 0$ *globally. Then every Lipschitz solution* $\overline{x}$ *is such that* $\overline{x} \in C^2$*. If* $L \in C^r$*,* $r > 2$ *then the solution* $\overline{x} \in C^r$.

Theorem 1.100 (Tonelli)**.** *Given the basic problem of Calculus of Variation on* $X = \mathrm{AC}([\mathrm{a}, \mathrm{b}])$ *with* L *continuous,* $v \longmapsto L(t, x, v)$ *convex and such that there exist* $\alpha > 0$*,* $\beta \in \mathbb{R}$*,* $p > 1$ *for which* $L(t, x, v) \geq \alpha |v|^p + \beta$*. Then the problem admits a solution* $\overline{x} \in \mathrm{AC}([\mathrm{a}, \mathrm{b}])$.

Theorem 1.101 (Clarke-Vinter (1985))**.** *Under the hypothesis of Theorem 1.100, if* L *doesn't depend from* t*, then all the solutions of the basic problem of Calculus of Variation are Lipschitz.*

Definition 1.102. The following functional, with the boundary conditions $x(a) = x_a$ and $x(b) = x_b$,

$$J(x) = \int_a^b L(t, x(t), \dot{x}(t))\,\mathrm{d}t$$

exhibits the *Laurent'ev phenomenon* if

$$\inf_{x \in \mathrm{AC}([a,b])} J(x) < \inf_{x \in \mathrm{Lip}([a,b])} J(x).$$

Remark 1.103. In the presence of Laurent'ev phenomenon, the usual finite elements numerical methods, setted in $\mathrm{W}^{1,\infty}(X) = \mathrm{Lip(X)}$ and used to minimize the basic problem of Calculus of Variation, can't reach the minimum.

1.10 Rectifiable Sets

Definition 1.104 (Rectifiable sets)**.** Let $A \subset \mathbb{R}^n$ be a $\mathscr{H}^k$-measurable set. We say that A is *countable k-rectifiable* if there exist countably many Lipschitz functions $u_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$A \subset \bigcup_{i=0}^{\infty} u_i(\mathbb{R}^k).$$

We say that A is *countably $\mathscr{H}^k$-rectifiable* if there exist countably many Lipschitz functions u_i : $\mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathscr{H}^k\left(A \setminus \bigcup_{i=0}^{\infty} u_i(\mathbb{R}^k)\right) = 0.$$

Finally, we say that A is *$\mathscr{H}^k$-rectifiable* if A is countably $\mathscr{H}^k$-rectifiable and $\mathscr{H}^k(A) < \infty$.

Remark 1.105. For $k = 0$ countably k-rectifiable and countably $\mathscr{H}^k$-rectifiable sets correspond to finite or countable sets, while $\mathscr{H}^k$-rectifiable sets correspond to finite sets. Moreover, rectifiable sets are stable under Lipschitz transformations.

Definition 1.106 (Rectifiable measures)**.** Let μ be a $\mathbb{R}^m$-valued Radon measure in $\mathbb{R}^n$. We say that μ is *k-rectifiable* if there exist a countably $\mathscr{H}^k$-rectifiable set S and a Borel function $\theta : S \to \mathbb{R}^m$ such that $\mu = \theta \mathscr{H}^k \llcorner S$.

The notion of locally $\mathscr{H}^k$-rectifiable set is the most important to us. Indeed, whenever A is countably $\mathscr{H}^k$-rectifiable set, the $\mathscr{H}^k \llcorner A$ is a regular Borel measure. However, $\mathscr{H}^k \llcorner A$ is a Radon measure if and only if A is locally $\mathscr{H}^k$-rectifiable. Therefore, is under the assumption of local $\mathscr{H}^k$-rectifiability on A that we have a natural identification between A and a Radon measure μ.

Example 1.107. Let Γ be a smooth curve in $\mathbb{R}^n$, that is $\Gamma = \gamma((a,b))$ for $\gamma : (a,b) \to \mathbb{R}^n$ smooth and injective. Given $t_0 \in (a,b)$, the tangent space to Γ at $x_0 = \gamma(t_0)$ is the line $\pi = \{s\gamma'(t_0) : s \in \mathbb{R}\}$.

Consider now Γ as a Radon measure, looking at $\mathscr{H}^1 \llcorner \Gamma$, and define the blow-ups $\mu_{x_0,\rho}$ of μ at x_0, setting

$$\mu_{x_0,\rho} = \frac{1}{\rho}(\Phi_{x_0,\rho})_\#(\mathscr{H}^1 \llcorner \Gamma) = \mathscr{H}^1 \llcorner \left(\frac{\Gamma - x_0}{\rho}\right),$$

with $\Phi_{x_0,\rho}(y) = (y - x_0)/\rho$, $y \in \mathbb{R}^n$. The fact that π is the tanget space to Γ at x_0 implies that

$$\mu_{x_0,\rho} \stackrel{*}{\rightharpoonup} \mathscr{H}^1 \llcorner \pi, \quad \text{as} \quad \rho \to 0^+.$$

Indeed, if $\varphi \in C_c^0(\mathbb{R}^n)$, from the push-foward of a Radon measure with $f : \mathbb{R}^m \to \mathbb{R}^n$ continuous and proper and a Borel measurable function $u : \mathbb{R}^m \to [0, \infty]$

$$\int_{\mathbb{R}^m} u \, \mathrm{d}(f_\# \mu) = \int_{\mathbb{R}^n} (u \circ f) \, \mathrm{d}\mu,$$

we find that

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu_{x_0,\rho} &= \frac{1}{\rho} \int_\Gamma \varphi\left(\frac{y - x_0}{\rho}\right) \mathrm{d}\mathscr{H}^1(y) = \frac{1}{\rho} \int_a^b \varphi\left(\frac{\gamma(t) - \gamma(t_0)}{\rho}\right) |\gamma'(t)| \, \mathrm{d}t \\
&= \int_{-(t_0 - a)/\rho}^{(b - t_0)/\rho} \varphi\left(\frac{\gamma(t_0 + \rho s) - \gamma(t_0)}{\rho}\right) |\gamma'(t_0 + \rho s)| \, \mathrm{d}s,
\end{aligned}$$

and

$$\lim_{\rho \to 0^+} \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu_{x_0,\rho} = \int_{\mathbb{R}} \varphi\big(s\gamma'(t_0)\big) |\gamma'(t_0)| \, \mathrm{d}s = \int_\pi \varphi \, \mathrm{d}\mathscr{H}^1.$$

Hence, if A is locally $\mathscr{H}^k$-rectifiable and $\mu = \mathscr{H}^k \llcorner A$, then for $\mathscr{H}^k$-a.e. $x \in A$ there exists a k-dimensional plane π_x in $\mathbb{R}^n$ such that the blow-ups $\mu_{x,\rho}$ of μ at x weak-$*$ converge to $\mathscr{H}^k \llcorner \pi_x$ as $\rho \to 0^+$, that is

$$\mathscr{H}^k \llcorner \left(\frac{A - x}{\rho}\right) \stackrel{*}{\rightharpoonup} \mathscr{H}^k \llcorner \pi_x, \quad \text{as } \rho \to 0^+.$$

Also the converse is also true: if μ is a Radon measure on $\mathbb{R}^n$ concentrated on a Borel set A and such that for every $x \in A$ there exists a k-dimensional plane π_x, such that the k-dimensional blow-ups of μ have the property that

$$\mu_{x,\rho} = \frac{(\varphi_{x,\rho})_\# \mu}{\rho^k} \stackrel{*}{\rightharpoonup} \mathscr{H}^k \llcorner \pi_x, \quad \text{as } \rho \to 0^+,$$

then A is locally $\mathscr{H}^k$-rectifiable and $\mu = \mathscr{H}^k \llcorner A$. The decomposition of rectifiable sets in Definition 1.104 allows us to prove the existence (in a measure-theoretic sense) of tangent space to rectifiable sets. Define $\Phi_{x,\rho} : \mathbb{R}^n \to \mathbb{R}^n$ as $\Phi_{x,\rho}(y) = (y - x)/\rho$, $y \in \mathbb{R}^n$, so that, if μ is a Radon measure on $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is Borel set, then

$$\frac{(\Phi_{x,\rho})_\# \mu(A)}{\rho^k} = \frac{\mu(x + \rho A)}{\rho^k}.$$

Theorem 1.108 (Existence of approximate tangent space). *If $A \subset \mathbb{R}^n$ is a locally $\mathscr{H}^k$-rectifiable set, then for $\mathscr{H}^k$-a.e. $x \in A$ there exists a unique k-dimensional plane π_x such that, as $\rho \to 0^+$,*

$$\frac{(\Phi_{x,\rho})_\#(\mathscr{H}^k \llcorner A)}{\rho^k} = \mathscr{H}^k \llcorner \left(\frac{A - x}{\rho}\right) \stackrel{*}{\rightharpoonup} \mathscr{H}^k \llcorner \pi_x, \tag{1.1}$$

that is

$$\lim_{\rho\to 0^+}\frac{1}{\rho^k}\int_A \varphi\left(\frac{y-x}{\rho}\right)\mathrm{d}\mathscr{H}^k(y)=\int_{\pi_x}\varphi\,\mathrm{d}\mathscr{H}^k,\quad \forall\varphi\in C_c^0(\mathbb{R}^n).$$

In particular, $\theta_k(\mathscr{H}^k \llcorner A)=1$ $\mathscr{H}^k$*-a.e. on* A *as*

$$\lim_{\rho\to 0^+}\frac{\mathscr{H}^k(A\cap B(x,\rho))}{\omega_k\rho^k}=1,\quad \textit{for } \mathscr{H}^k\textit{-a.e. } x\in A.$$

Remark 1.109. If a k-dimensional plane π_x satisfies Equation (1.1), then we set $\pi_x = T_xA$ and name it the *approximate tangent space to* A *at* x. The set of points $x\in A$ such that Equation (1.1) holds true depends only on the Radon measure $\mu=\mathscr{H}^k\llcorner A$. It is a locally $\mathscr{H}^k$-rectifiable set in $\mathbb{R}^n$, which is left unchanged if we modify A on and by $\mathscr{H}^k$-null sets.

Lemma 1.110. *If* $A=u(E)$ *is a* k*-dimensional regular Lipschitz image in* $\mathbb{R}^n$ *and* $z\in E$*, then*

$$T_xA=\nabla u(z)(\mathbb{R}^k),\quad x=u(z).$$

Proposition 1.111 (Locality of approximate tangent spaces)**.** *If* A_1 *and* A_2 *are locally* $\mathscr{H}^k$*-rectifiable sets in* $\mathbb{R}^n$*, then for* $\mathscr{H}^k$*-a.e.* $x\in A_1\cap A_2$*,*

$$T_xA_1=T_xA_2.$$

Example 1.112 (Tangent space to a graph)**.** If $u:\mathbb{R}^{n-1}\to\mathbb{R}$ is a Lipschitz function, and we define $f:\mathbb{R}^{n-1}\to\mathbb{R}^n$ as $f(z)=(z,u(z))$, $z\in\mathbb{R}^{n-1}$, then $\Gamma=f(\mathbb{R}^{n-1})$ is locally $\mathscr{H}^{n-1}$-rectifiable and, for a.e. $z\in\mathbb{R}^{n-1}$,

$$T_{f(x)}\Gamma=\nu(z)^\perp,\quad \nu(z)=(\nabla' u(z),1).$$

CHAPTER 2

Sets of Finite Perimeter

Introduced by R. Caccioppoli in [Caccioppoli(1927)], the theory of sets of finite perimeter is closely connected to the theory of BV function (Chapter 3): the set $A \subset \Omega$ has finite perimeter in Ω, indicated by $P(A;\Omega)$, if and only if $\chi_A \in \mathrm{BV}(\Omega)$. In this case, $P(A;\Omega)$ coincides with $|D\chi_A|(\Omega)$, the total variation in Ω of the distributional derivative of χ_A. Essentially, a set of Finite Perimeter is a set whose boundary is measurable and has a (at least locally) finite measure. It is also called a *Caccioppoli set*.

The key point to investigate is when the divergence of a vector field $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, bounded and measurable, is a totally finite signed measure μ, i.e. $\operatorname{div}\varphi = \mu$, and what is the sense of the Gauss-Green theorem. This chapter recalls many results from [Ambrosio *et al.*(2000), Maggi(2012), Giusti(1977)].

2.1 The Gauss-Green Theorem

Open sets with C^1-boundary

Definition 2.1. Let A be an open set in $\mathbb{R}^n$ and let $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 1$. We say that A has a C^k-*boundary* (or smooth boundary if $k = \infty$) if for every $x \in \partial A$ there exist $\psi \in C^k(B(x,\rho))$, $\rho > 0$, with $\nabla\psi(y) \neq 0$ for every $y \in B(x,\rho)$ and

$$B(x,\rho) \cap A = \Big\{ y \in B(x,\rho) : \psi(y) < 0 \Big\} \quad \text{and} \quad B(x,\rho) \cap \partial A = \Big\{ y \in B(x,\rho) : \psi(y) = 0 \Big\}.$$

Definition 2.2 (Outer normal)**.** The outer normal ν_A to A is defined locally as

$$\nu_A = \frac{\nabla\psi(y)}{|\nabla\psi(y)|}, \quad \forall y \in B(x,\rho) \cap \partial A.$$

This definition is independent of the choice of ψ and ρ, therefore ν_A can be considered as a vector field on the whole ∂A, with $\nu_A \in C^{k-1}(\partial A; S^{n-1})$.

Remark 2.3. If A is an open set with C^1-boundary, then $\mathcal{H}^{n-1} \llcorner \partial A$ is a Radon measure on $\mathbb{R}^n$.

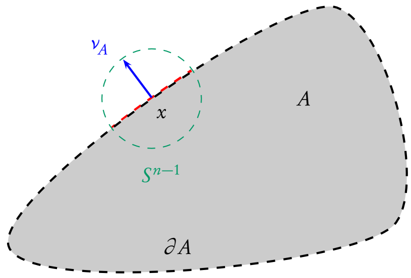


Figure 3: The outer normal ν_A.

The elementary Gauss-Green formula

$$\int_{\mathbb{R}^n} \varphi \nabla u = -\int_{\mathbb{R}^n} u \nabla \varphi, \quad \forall u \in C^1(\mathbb{R}^n), \varphi \in C_c^1(\mathbb{R}^n),$$

motivates the introduction of the *distributional gradient* Du of a function $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ as the linear functional $Du : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}^n$,

$$\langle Du, \varphi \rangle = -\int_{\mathbb{R}^n} u \nabla \varphi, \quad \varphi \in C_c^\infty(\mathbb{R}^n).$$

Whenever Du is representable as integration of the test function φ against an $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ vector field, that is, if there exists a vector field $T \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} u \nabla \varphi = -\int_{\mathbb{R}^n} \varphi T, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n),$$

we say that u has a *weak gradient* on $\mathbb{R}^n$.

Lemma 2.4 (Vanishing weak gradient)**.** *Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, if A is open and connected with*

$$\int_{\mathbb{R}^n} u \nabla \varphi = 0, \quad \forall \varphi \in C_c^\infty(A),$$

then there exists $c \in \mathbb{R}$ such that $u = c$ a.e. in A.

Theorem 2.5 (Gauss-Green)**.** *If A is an open set with C^1-boundary with ν_A the outer normal, then for every $\varphi \in C_c^1(\mathbb{R}^n)$,*

$$\int_A \nabla \varphi(x)\,\mathrm{d}x = \int_{\partial A} \varphi \nu_A \,\mathrm{d}\mathcal{H}^{n-1}.$$

Equivalently, the divergence theorem holds true:

$$\int_A \operatorname{div} T(x)\,\mathrm{d}x = \int_{\partial A} T \cdot \nu_E \,\mathrm{d}\mathcal{H}^{n-1}, \quad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

Open sets with almost C^1-boundary

Definition 2.6. An open set $A \subset \mathbb{R}^n$ has *almost C^1-boundary* if there exists a decomposition $\partial A = M \cup M_0$, with M_0 a closed set and

$$\mathcal{H}^{n-1}(M_0) = 0$$

such that, for every $x \in M = \partial A \setminus M_0$, there exists $\rho > 0$ and $\psi \in C^1(B(x,\rho))$ with the property that

$$B(x,\rho) \cap A = \Big\{ y \in B(x,\rho) : \psi(y) < 0 \Big\},$$
$$B(x,\rho) \cap \partial A = B(x,\rho) \cap M = \Big\{ y \in B(x,\rho) : \psi(y) = 0 \Big\}.$$

We call the C^1-hypersurface M the *regular part* of ∂A and the outer normal to A is defined as a continuous vector field $\nu_A \in C^0(M; S^{n-1})$, through the local representation

$$\nu_A(y) = \frac{\nabla \psi(y)}{|\nabla \psi(y)|}, \quad \forall y \in B(x,\rho) \cap M.$$

Theorem 2.7 (Gauss-Green). *If A is an open set in $\mathbb{R}^n$ with almost C^1-boundary, and M is the regular part of ∂A, then for every $\varphi \in C_c^1(\mathbb{R}^n)$*

$$\int_A \nabla \varphi = \int_M \varphi \nu_A \, \mathrm{d}\mathscr{H}^{n-1}.$$

Surfaces

Definition 2.8. If $A \subset \mathbb{R}^n$ is a k-dimensional C^1-surface and $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ we shall say that

- T is *tangential* to A if $T(x) \in T_x A$ (tangent space) for every $x \in A$;
- T is *normal* to A if $T(x) \in (T_x A)^\perp$ for every $x \in A$.

Theorem 2.9 (Gauss-Green). *If $M \subset \mathbb{R}^n$ is a C^2-hypersurface with boundary Γ, then there exist a normal vector field $\mathbf{H}_M \in C^0(M; \mathbb{R}^n)$ to M and a normal vector field $\nu_\Gamma^M \in C^1(\Gamma; S^{n-1})$ to Γ such that*

$$\int_M \nabla^M \varphi \, \mathrm{d}\mathscr{H}^{n-1} = \int_M \varphi \mathbf{H}_M \, \mathrm{d}\mathscr{H}^{n-1} + \int_\Gamma \varphi \nu_\Gamma^M \, \mathrm{d}\mathscr{H}^{n-2},$$

for every $\varphi \in C_c^1(\mathbb{R}^n)$. Moreover, if $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ is normal to M, then

$$T \cdot \nu_\Gamma^M = 0 \quad \text{on } \Gamma.$$

Remark 2.10. The vector field $\mathbf{H}_M$ is called the *mean curvature vector* to A. The definition of the *scalar mean curvature* $H_M : M \to \mathbb{R}$ of M depends on the mean curvature vector and the explicit choice of a unit normal vector field $\nu_M : M \to S^{n-1}$ to M through the formula

$$\mathbf{H}_M = H_M \nu_M.$$

If there exists a continuous unit normal vector field ν_M to M, then ν_M is an *orientation* of M and M is orientable and H_M can be assumed continuous on M.

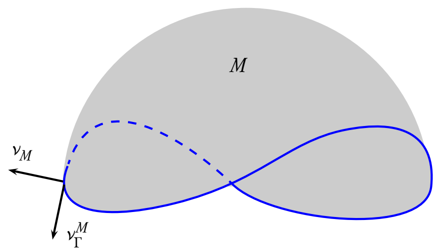


Figure 4: Normals the boundary Γ of A induced through the tangential divergence theorem on A.

Theorem 2.11 (Divergence theorem on surfaces)**.** *Given a vector field* $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, *we define the* tangential divergence *of* T *on* M *by the formula*

$$\operatorname{div}^M T = \operatorname{div} T - (\nabla T \nu_M)\cdot \nu_M = \operatorname{trace}(\nabla^M T).$$

where $\nu_M : M \to S^{n-1}$ *is any unit normal vector field to* M.

Remark 2.12. Discontinuously switching ν_M to $-\nu_M$ leaves $\operatorname{div}^M T$ unchanged. Hence, it is always $\operatorname{div}^M T \in C^0(M)$, even if M is not orientable.

2.2 Area Formula

Let $u : \mathbb{R}^n \to \mathbb{R}^m$ an injective Lipschitz function with $1 \le n \le m$ and $A \in \mathbb{R}^n$. By Theorem 1.91, $u(A)$ is at most n-dimensional in $\mathbb{R}^m$. The area formula permits to express $\mathcal{H}^n(u(A))$ in terms of integration over A of the *Jacobian* of u, defined as $Ju : \mathbb{R}^n \to [0,\infty]$,

$$Ju(x) = \begin{cases} \sqrt{\det(\nabla u(x) * \nabla u(x))} & \text{if } u \text{ is differentiable at } x; \\ +\infty & \text{if } u \text{ is not differentiable at } x \end{cases}$$

Remark 2.13. The set $\{x \in \mathbb{R}^n : Ju(x) < \infty\}$ coincides with the set of points $x \in \mathbb{R}^n$ at which u is differentiable so, from Radamacher's theorem, $\{Ju < \infty\}$ has full Lebesgue measure in $\mathbb{R}^n$.

Theorem 2.14 (Area formula for injective maps)**.** *If* $u : \mathbb{R}^n \to \mathbb{R}^m$, *for* $1 \le n \le m$, *is a injective Lipschitz function and* $A \subset \mathbb{R}^n$ *is Lebesgue measurable, then*

$$\mathcal{H}^n(u(A)) = \int_A Ju(x)\,\mathrm{d}x,$$

and $\mathcal{H}^n \llcorner u(\mathbb{R}^n)$ *is a Radon measure on* $\mathbb{R}^m$.

Lemma 2.15. *If* A *is a Lebesgue measurable set in* $\mathbb{R}^n$ *and* $u : \mathbb{R}^n \to \mathbb{R}^m$, $1 \le n \le m$, *is a Lipschitz function, then* $u(A)$ *is* $\mathcal{H}^n$*-measurable in* $\mathbb{R}^m$.

Theorem 2.16. *If $u : \mathbb{R}^n \to \mathbb{R}^m$, $1 \le n \le m$, is a Lipschitz function and $A = \{x \in \mathbb{R}^n; Ju = 0\}$, then*

$$\mathscr{H}^n(u(A)) = 0,$$

So the singular set $\{Ju = 0\}$ is mapped by u into an $\mathscr{H}^n$-negligible set.

Let A be a locally $\mathscr{H}^k$-rectifiable set in $\mathbb{R}^n$ and let $x \in A$ be such that the approximate tangent space T_xA exists. As in the C^1-case, we say that $u : \mathbb{R}^n \to \mathbb{R}^m$ is *tangentially differentiable with respect to A at x* if the restriction of u to $x + T_xA$ is differentiable at x.

Theorem 2.17. *If A is a locally $\mathscr{H}^k$-rectifiable set, and $u : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map, then $\nabla^A u(x)$ exists at $\mathscr{H}^k$-a.e. $x \in A$.*

Theorem 2.18 (Area formula on rectifiable sets). *If A is a locally $\mathscr{H}^k$-rectifiable set and $u : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map with $1 \le k \le m$, then*

$$\int_{\mathbb{R}^m} \mathscr{H}^0(A \cap \{u = y\})\, \mathrm{d}\mathscr{H}^k(y) = \int_A J^A u \, \mathrm{d}\mathscr{H}^k,$$

where $\{u = y\} = \{x \in \mathbb{R}^n : u(x) = y\}$. In particular, if u is injective in A, then

$$\mathscr{H}^k(u(A)) = \int_A J^A u \, \mathrm{d}\mathscr{H}^k.$$

Definition 2.19. Let M be a k-dimensional C^1-surface in $\mathbb{R}^n$ and let $x \in M$. A function $u : \mathbb{R}^n \to \mathbb{R}^m$ is *tangentially differentiable with respect to M at x*, if there exists a linear function $\nabla^M u(x) \in \mathbb{R}^m \otimes T_xM$ such that, uniformly on $\{v \in T_xM : |v| = 1\}$,

$$\lim_{h \to 0} \frac{u(x + hv) - u(x)}{h} = \nabla^M u(x)v.$$

In other words, the restriction of u to $x + T_xM$ is differentiable at x. The *tangential Jacobian* of u with respect to A at x is then defined by

$$J^M u(x) = \sqrt{\det(\nabla^M u(x) * \nabla^M u(x))}.$$

Theorem 2.20. *If $M \subset \mathbb{R}^n$ is a k-dimensional C^1-surface and $u \in C^1(\mathbb{R}^n; \mathbb{R}^m)$, with $m \ge k$, is injective, then*

$$\mathscr{H}^k(u(M)) = \int_M J^M u \, \mathrm{d}\mathscr{H}^k.$$

2.3 Set of Finite Perimeter

Definition 2.21 (Set of finite perimeter). Let A be a Lebesgue measurable subset in $\mathbb{R}^n$. We say that A is a *set of locally finite perimeter* in $\mathbb{R}^n$ if for every compact set $K \subset \mathbb{R}^n$ we have

$$\sup\left\{ \int_A \operatorname{div} \varphi(x)\, \mathrm{d}x : \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n),\ \operatorname{supp}\varphi \subset K,\ \sup_{\mathbb{R}^n} |\varphi| \le 1 \right\} < \infty.$$

If this quantity is bounded independently of K, then we say that A is a *set of finite perimeter* in $\mathbb{R}^n$.

Definition 2.22. We denote the *relative perimeter* of $A \in \Omega \subset \mathbb{R}^n$ the variation of $\chi_A \in \Omega$ by

$$P(A;\Omega) = |\mu_A|(\Omega) = \sup\left\{ \int_A \operatorname{div}\varphi(x)\,\mathrm{d}x : \varphi \in C_c^1(\Omega;\mathbb{R}^n), \|\varphi\|_\infty \leq 1 \right\}$$

and the *total perimeter* of A by

$$P(A) = |\mu_A|(\mathbb{R}^n).$$

Proposition 2.23. *If A is a Lebesgue measurable set in $\mathbb{R}^n$, then A is a set of locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure μ_A on $\mathbb{R}^n$ such that*

$$\int_A \nabla\varphi = \int_{\mathbb{R}^n} \varphi\,\mathrm{d}\mu_A, \quad \forall \varphi \in C_c^1(\mathbb{R}^n). \tag{2.1}$$

Moreover, A is a set of finite perimeter if and only if $|\mu_A|(\mathbb{R}^n) < \infty$.

Proof. Let A be a set of locally finite perimeter in $\mathbb{R}^n$, and consider the linear functional

$$L : C_c^1(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R} \text{ defined by } \langle L,\varphi\rangle = \int_A \operatorname{div}\varphi(x)\,\mathrm{d}x.$$

For every compact set $K \in \mathbb{R}^n$, there exists $C(K) \in \mathbb{R}$ such that

$$|\langle L,\varphi\rangle| \leq C(K)\sup_{\mathbb{R}^n}|\varphi|, \quad \text{whenever } \operatorname{supp}\varphi \subset K.$$

Hence, L can be extended by density to a bounded continuous linear functional on $C_c^0(\mathbb{R}^n;\mathbb{R}^n)$, and the existence of μ_A follows by Riesz Theorem. Clearly, if A is a set of finite perimeter then $\mu_A(\mathbb{R}^n) < \infty$. The converse implications are trivial. Indeed if $K \in \mathbb{R}^n$ is compact, $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ with $|\varphi| \leq 1$ on $\mathbb{R}^n$ and $\operatorname{supp}\varphi \subset K$, then by Equation (2.1) we have

$$\int_A \operatorname{div}\varphi(x)\,\mathrm{d}x \leq |\mu_A|(K).$$

□

Remark 2.24. Every set of Lebesgue measure zero is of finite perimeter, and has perimeter zero.

Remark 2.25. The class of sets of finite perimeter in Ω contains all the sets A with C^1-boundary inside Ω such that $\mathscr{H}^{n-1}(\Omega \cap \partial A) < \infty$. Indeed by the classical Gauss-Green Theorem, for these open sets A with C^1-boundary with the outer normal $\nu_A \in C^0(\partial A; S^{n-1})$ we have

$$\int_A \nabla\varphi\,\mathrm{d}x = \int_{\Omega\cap\partial A} \varphi\nu_A\,\mathrm{d}\mathscr{H}^{n-1}, \quad \forall\varphi \in C_c^1(\mathbb{R}^n),$$

or equivalently,

$$\int_A \operatorname{div}\varphi\,\mathrm{d}x = \int_{\Omega\cap\partial A} \langle\nu_A,\varphi\rangle\,\mathrm{d}\mathscr{H}^{n-1}, \quad \forall\varphi \in C_c^1(\Omega),$$

and thus A is a set of locally finite perimeter with

$$\mu_A = \nu_A\mathscr{H}^{n-1} \llcorner \partial A, \qquad |\mu_A| = \mathscr{H}^{n-1} \llcorner \partial A$$
$$P(A;\Omega) = \mathscr{H}^{n-1}(\Omega \cap \partial A), \qquad P(A) = \mathscr{H}^{n-1}(\partial A)$$

for every $\Omega \subset \mathbb{R}^n$. The Radon measure μ_A is called the *Gauss-Green measure* on A.

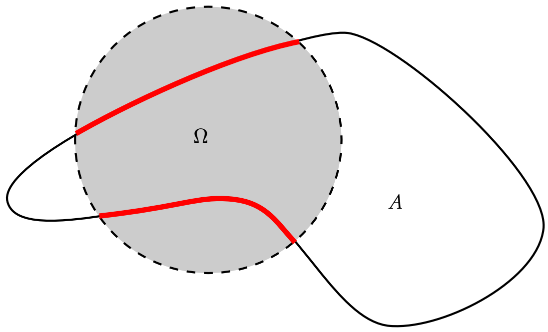


Figure 5: The perimeter $P(A;\Omega)$ of A relative to Ω is the $(n-1)$-dimensional measure of the intersection of the (reduced) boundary of A with Ω, marked in red.

Property 2.26. We summarize some useful properties:

- *Invariant*: $P(A)$ of A is invariant by modifications of A on and/or by a set of measure zero, although these modifications may widely affect the size of its topological boundary of A;
- *Scaling and translation*: if $\lambda > 0$, $\mathbf{x} \in \mathbb{R}^n$ and A is a set of finite perimeter in $\mathbb{R}^n$, then $x + \lambda A$ is a set of finite perimeter with $P(x+\lambda A) = \lambda^{n-1} P(A)$;
- *Complement*: if A is a set of locally finite perimeter, then $\mathbb{R}^n \setminus A$ is a set of locally finite perimeter with $\mu_{\mathbb{R}^n\setminus A} = -\mu_A$ and $P(A) = P(\mathbb{R}^n \setminus A)$;

Definition 2.27 (Local convergence)**.** Given Lebesgue measurable sets $\{A_h\}_{h\in\mathbb{R}^n}$, we say that A_h *locally converges* to A, and write $A_h \xrightarrow{\text{loc}} A$, if

$$\lim_{h\to\infty} \left| K \cap (A \Delta A_h) \right| = 0, \quad \forall K \subset \mathbb{R}^n \text{ compact},$$

where Δ here is the symmetric difference operation in set theory. Moreover, we say that A_h *converges* to A, and write $A_h \to A$, if

$$\lim_{h\to\infty} |A \Delta A_h| = 0.$$

Indeed, if Ω is a set of locally finite perimeter in $\mathbb{R}^n$, we find

$$P(A;\Omega) = \sup\left\{ \int_A \operatorname{div}\varphi(x)\,\mathrm{d}x : \varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \sup_{\mathbb{R}^n} |\varphi| \le 1 \right\}.$$

Proposition 2.28 (Lower semicontinuity of perimeter)**.** *If $\{A_h\}_{h\in\mathbb{N}}$ is a sequence of sets of locally finite perimeter in $\mathbb{R}^n$, with*

$$A_h \xrightarrow{loc} A, \quad \limsup_{h\to\infty} P(A_h;K) < \infty,$$

for every compact set $K \in \mathbb{R}^n$, then A is of locally finite perimeter in $\mathbb{R}^n$, $\mu_{A_h} \overset{*}{\rightharpoonup} \mu_A$ and, for every open set $A \subset \mathbb{R}^n$ we have

$$P(A;\Omega) \le \liminf_{h\to\infty} P(A_h;\Omega).$$

Proposition 2.29. *If A is a set of locally finite perimeter in $\mathbb{R}^n$, then*

$$\operatorname{supp}\mu_A = \left\{x \in \mathbb{R}^n : 0 < |A \cap B(x,\rho)| < \omega_n \rho^n, \forall \rho > 0\right\} \subset \partial A.$$

Moreover, there exists a Borel set Ω such that $|A \Delta \Omega| = 0$ and $\operatorname{supp}\mu_\Omega = \partial\Omega$.

Theorem 2.30. *For any set A of finite perimeter in Ω the distributional derivative $D\chi_A$ is an $\mathbb{R}^n$-valued finite measure in Ω. Moreover, $P(A;\Omega) = |D\chi_A|(\Omega)$ and a generalized Gauss-Green formula holds:*

$$\int_A \operatorname{div}\varphi \, \mathrm{d}x = \int_\Omega \chi_A \operatorname{div}\varphi \, \mathrm{d}x - \int_\Omega \langle \nu_A, \varphi \rangle \, \mathrm{d}|D\chi_A|, \quad \forall \varphi \in [C_c^1(\Omega)]^n$$

where $D\chi_A = \nu_A |D\chi_A|$ is the polar decomposition of $D\chi_A$.

Remark 2.31. A Lebesgue measurable set $A \subset \mathbb{R}^n$ is a set of locally finite perimeter if and only if the distributional gradient $D\chi_A$ of $\chi_A \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ can be represented as the integration with respect to the Radon measure $-\mu_A$.

Remark 2.32. An open set A with Lipschitz or polyhedral boundary is a locally finite perimeter, with $P(A;\Omega) = \mathscr{H}^{n-1}(\Omega \cap \partial A)$ whenever $\Omega \subset \mathbb{R}^n$. Moreover, if A is bounded, then A is of finite perimeter. In particular, convex sets are of locally finite perimeter, while bounded convex sets are of finite perimeter.

Lemma 2.33. *If A and B are sets of (locally) finite perimeter in $\mathbb{R}^n$, then $A \cup B$ and $A \cap B$ are sets of (locally) finite perimeter in $\mathbb{R}^n$, and, for $A \subset \mathbb{R}^n$ open,*

$$P(A \cup B;\Omega) + P(A \cap B;\Omega) \leq P(A;\Omega) + P(B;\Omega).$$

Proof. We can consider u_n, v_n smooth functions such that $u_n \to \chi_A$, $v_n \to \chi_B$, and, as $n \to \infty$,

$$\int_\Omega |\nabla u_n(x)| \, \mathrm{d}x \to P(A;\Omega), \qquad \int_\Omega |\nabla v_n(x)| \, \mathrm{d}x \to P(B;\Omega).$$

Then,

$$u_n \vee v_n = \max\{u_n, v_n\} \to \chi_{A\cup B} \quad \text{and} \quad u_n \wedge v_n = \min\{u_n, v_n\} \to \chi_{A\cap B},$$

and using the lower semicontinuity of the total variation we find

$$P(A \cup B;\Omega) + P(A \cap B;\Omega) \leq \liminf_{n\to\infty} \int_\Omega |\nabla(u_n \vee v_n)| + |\nabla(u_n \wedge v_n)| \, \mathrm{d}x.$$

The inequality follows from

$$\int_\Omega |\nabla(u_n \vee v_n)(x)| + |\nabla(u_n \wedge v_n)(x)| \, \mathrm{d}x = \int_\Omega |\nabla u_n(x)| + |\nabla v_n(x)| \, \mathrm{d}x \to P(A;\Omega) + P(B;\Omega). \quad \square$$

2.4 Reduced boundary

Definition 2.34 (Reduced boundary)**.** The *reduced boundary* $\partial^* A$ of a set of locally finite perimeter $A \in \mathbb{R}^n$ is the set of those $x \in \operatorname{supp} \mu_A$ such that the limit

$$\lim_{\rho \to 0^+} \frac{\mu_A(B(x,\rho))}{|\mu_A|(B(x,\rho))}, \quad \text{exists and belongs to } S^{n-1}.$$

We may define a Borel function $\mu_A : \partial^* A \to S^{n-1}$ by setting

$$\nu_A(x) = \lim_{\rho \to 0^+} \frac{\mu_A(B(x,\rho))}{|\mu_A|(B(x,\rho))}, \quad x \in \partial^* A.$$

We call ν_A the *measure-theoretic outer unit normal to* A.

Remark 2.35. By the Bezicovič Theorem 1.55, we have $\mu_A = \nu_A |\mu_A| \llcorner \partial^* A$, so that the distributional Gauss-Green Theorem takes the form

$$\int_A \nabla \varphi = \int_{\partial^* A} \varphi \nu_A \, \mathrm{d}|\mu_A|, \quad \forall \varphi \in C_c^1(\mathbb{R}^n).$$

Example 2.36. If A is an open set with C^1 boundary, then $\partial^* A = \partial A$ and the measure-theoretic outer unit normal coincides with the classical notion of outer unit normal.

Remark 2.37. If A is a set of locally finite perimeter and $|A \Delta \Omega| = 0$, then $\mu_A = \mu_\Omega$ and therefore $\partial^* A = \partial^* \Omega$: the reduced boundary $\partial^* A$ is uniquely determined by the Gauss-Green measure μ_A of A.

Remark 2.38. By definition, $\partial^* A \subset \operatorname{supp} \mu_A$ while, by Proposition 2.29, $\operatorname{supp} \mu_A \subset \partial A$. So $\partial^* A \subset \operatorname{supp} \mu_A \subset \partial A$. Hence, the reduced boundary is always a subset of the topological boundary. In fact, the Gauss-Green measure μ_A is always concentrated on $\partial^* A$, and hence on $\overline{\partial^* A}$. By definition of support, $\operatorname{supp} \mu_A \subset \overline{\partial^* A}$, and therefore

$$\operatorname{supp} \mu_A = \overline{\partial^* A}.$$

Therefore, up to modification on sets of measure zero,

$$\overline{\partial^* A} = \partial A.$$

Example 2.39. If $A \subset \mathbb{R}^2$ is a square with sides parallel to the coordinate axes, then the limit $\nu_A(x)$ exists for every $x \in \partial A$. However $|\nu_A| = 1$ if and only if x is not a vertex of A: indeed, if x is a vertex, then $\nu_A(x) = |(e_1 + e_2)/2| < 1$. Thus, $\partial^* A = \partial A$ minus the four vertexes of A.

Considering the blow-up $A_{x,\rho}$ of A:

$$A_{x,\rho} = \frac{A - x}{\rho} = \phi_{x,\rho}(A), \quad x \in \mathbb{R}^n, \rho > 0,$$

where, as usual, $\phi_{x,\rho}(y) = (y-x)/\rho$, $y \in \mathbb{R}^n$, by Lebesgue's point Theorem we have

$$x \in A^{(1)} \qquad \text{if and only if } A_{x,\rho} \xrightarrow{\text{loc}} \mathbb{R}^n, \qquad \text{as } \rho \to 0^+$$
$$x \in A^{(0)} \qquad \text{if and only if } A_{x,\rho} \xrightarrow{\text{loc}} \emptyset, \qquad \text{as } \rho \to 0^+.$$

Theorem 2.40 (Tangential properties of the reduced boundary)**.** *If A is a set of locally finite perimeter in $\mathbb{R}^n$, and $x \in \partial^* A$, then*

$$A_{x,\rho} \xrightarrow{loc} H_x = \left\{ y \in \mathbb{R}^n : y \cdot \nu_A(x) \le 0 \right\}, \quad \textit{as } \rho \to 0^+.$$

Similarly, if $\pi_x = \partial H_x = \nu_A(x)^\perp$, then, as $\rho \to 0^+$,

$$\mu_{A_{x,\rho}} \overset{*}{\rightharpoonup} \nu_A(x) \mathcal{H}^{n-1} \pi_x, \quad |\mu_{A_{x,\rho}}| \overset{*}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \pi_x.$$

Remark 2.41. As consequences, a set of locally finite perimeter has density one-half on its reduced boundary while the relative perimeter of A inside balls $B(x,\rho)$ centered at $x \in \partial^* A$ is asymptotic to the measure of a $(n-1)$-dimensional ball of radius ρ as $\rho \to 0^+$.

Corollary 2.42. *If A is a set of locally finite perimeter and $x \in \partial^* A$, then*

$$\lim_{\rho \to 0^+} \frac{A \cap B(x,\rho)}{\omega_n \rho^n} = \frac{1}{2} \quad \textit{and} \quad \lim_{\rho \to 0^+} \frac{P(A; B(x,\rho))}{\omega_n \rho^n} = 1.$$

In particular, $\partial^* A \subset A^{(1/2)}$, the set of points of density one-half of A.

Theorem 2.43 (De Giorgi's structure theorem)**.** *If A is a set of locally finite perimeter in $\mathbb{R}^n$, then the Gauss-Green measure μ_A of A satisfies*

$$\mu_A = \nu_A \mathcal{H}^{n-1} \llcorner \partial^* A,$$
$$|\mu_A| = \mathcal{H}^{n-1} \llcorner \partial^* A,$$

and the generalized Gauss-Green formula holds true:

$$\int_A \nabla \varphi = \int_{\partial^* A} \varphi \nu_A \, \mathrm{d}\mathcal{H}^{n-1}, \quad \forall \varphi \in C_c^1(\mathbb{R}^n).$$

Moreover, there exist countably many C^1-hypersurface M_h in $\mathbb{R}^n$, compact sets $K_h \subset M_h$, and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$ such that

$$\partial^* A = F \cup \bigcup_{h \in \mathbb{N}} K_h,$$

and, for every $x \in K_h$, $\nu_A(x)^\perp = T_x M_h$, the tangent space to M_h at x.

2.5 Perimeter problems in Calculus of Variations

Isoperimetric problem

The isoperimetric problem has its roots in *Aeneid* by the Roman poet *Virgil*. According to the myth, Dido was the eldest daughter of Belus, king of Tyre, and she got married with Acerbas (also called Sichaeus), a priest of Hercules, the richest Phoenician. One day, Pygmalion, Dido's brother, blinded by lust to own the royal treasure, surprised and murdered Sichaeus, while he was sacrificing to the gods. For a long time, Pygmalion hid this murder. But the ghost of Sichaeus, deprived by the honours of a tomb, appeared in a dream to Dido, showing her the altar where he was assassinated. Then, he encouraged her to escape with the Phoenician treasures. Dido left Tyre with a large following and began a long pilgrimage, whose main stages were Cyprus and Malta. Juno, the Queen of gods and Dido's protector, promised them a new land to build their new city in the point where they would have found an horse's skull, under the sand. After having landed the coast of Libya, Dido obtained from Iarba, the Libian King, the permission to settle there. According to King's will, she would have owned *as much land as an ox-hide could have contained*. Astutely, Dido chose a peninsula and cut the skin of a bull into many thin strips and put them around the land: the future Carthage. She was able to occupy a territory of about twenty-two stages (a stage is equivalent to about 185.27 m^2) . For all these reasons, the old name of Carthage is *Birsa*, which in Greek means *ox-hide* and in Phoenician *fortress*. This problem can be reformulated as a basic problem of Calculus of Variation (see [Marigonda(2012)] for more details):

$$J(x) = -A(x) = -\int_a^b x(t)\,dt, \quad \text{such that} \quad \int_a^b \sqrt{1+\dot{x}(t)}\,dt = \ell > b-a,$$

with $x(a) = x(b) = 0$. If $\overline{x}$ is a solution, we consider the extremal points of the Lagrangian function

$$\mathscr{L}(t,x,v) = -\lambda_0 x + \lambda\sqrt{1+v^2}.$$

If $\ell > b-a$, then $\lambda_0 \neq 0$ so we can take $\lambda_0 = 1$ and the Euler's equations become

$$\frac{d}{dt}\left[-\lambda\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right] = -1, \quad \text{or} \quad \frac{-\lambda\dot{x}}{\sqrt{1+\dot{x}^2}} = -t + C.$$

Raising the square, we obtain

$$\dot{x} = \frac{c-t}{\sqrt{\lambda^2-(c-t)^2}},$$

from which $x(t) = \sqrt{\lambda^2-(c-t)^2} + K$ so $(x-K)^2 + (t-c)^2 = \lambda^2$. Hence, given some curves with equal lengths, the circumference described the largest area. So Dido's problem is closely related to the *isoperimetric problem*.

Definition 2.44 (Euclidean isoperimetric problem)**.** We call the *Eucludean isoperimetric problem*

$$\inf\left\{P(A):|A|=m\right\},\quad m>0.$$

Theorem 2.45 (Euclidean isoperimetric inequality)**.** *If A is a Lebesgue measurable set in $\mathbb{R}^n$ with $|A|<\infty$, then*

$$P(A)\geq n\omega_n^{1/n}|A|^{(n-1)/n}.$$

Equality holds if and only if $|A\Delta B(x,\rho)|=0$ for some $x\in\mathbb{R}^n$, $\rho>0$.

Remark 2.46. Theorem 2.45 is false in dimension one: the half-line $A=(0,\infty)$ has finite perimeter in $\mathbb{R}$ but both A and $\mathbb{R}\setminus A$ have infinite measure.

Definition 2.47 (Steiner symmetrization)**.** Let us decompose $\mathbb{R}^n$, $n\geq 2$, as the product $\mathbb{R}^{n-1}\times\mathbb{R}$, with the projections $\mathbf{p}:\mathbb{R}^n\to\mathbb{R}^{n-1}$ and $\mathbf{q}:\mathbb{R}^n\to\mathbb{R}$, so that $x=(\mathbf{p}x,\mathbf{q}x)$ for $x\in\mathbb{R}^n$ (in particular, $\mathbf{q}x=x_n$). With every $z\in\mathbb{R}^{n-1}$ we associate the vertical slice $A_z\subset\mathbb{R}$ of A defined as

$$A_z=\left\{t\in\mathbb{R}:(z,t)\in A\right\},$$

and define the Steiner symmetrization A^s of A as

$$A^s=\left\{x\in\mathbb{R}^n:|\mathbf{q}x|\leq\frac{\mathscr{L}^1(A_{\mathbf{p}x})}{2}\right\}.$$

By Fubini's Theorem, $|A|=|A^s|$ and diameters are decreased under Steiner symmetrization.

Theorem 2.48. *If A is a set of finite perimeter in $\mathbb{R}^n$, with $|A|<\infty$, then A^s is a set of finite perimeter in $\mathbb{R}^n$ with*

$$P(A^s)\leq P(A), \tag{2.2}$$

and, in fact, whenever Ω is an open set in $\mathbb{R}^{n-1}$, $P(A^s;\Omega\times\mathbb{R})\leq P(A;\Omega\times\mathbb{R})$. Moreover,

- *if equality holds in Equation* (2.2)*, then, for a.e. $z\in\mathbb{R}^{n-1}$, the vertical slice A_z is equivalent to an interval;*
- *if A is equivalent to a convex set, then equality holds in* (2.2) *if and only if there exists $c\in\mathbb{R}$ such that A is equivalent to A^s+ce_n.*

Proposition 2.49 (Slicing perimeter by lines)**.** *If A is a set of locally finite perimeter in $\mathbb{R}^n$, then, for a.e. $z\in\mathbb{R}^{n-1}$, the vertical slice A_z is a set of locally finite perimeter in $\mathbb{R}$, and, for $I\subset\mathbb{R}$ bounded and open, and $H\subset\mathbb{R}^{n-1}$ compact,*

$$\int_H P(A_z;I)\,\mathrm{d}z\leq P(A;H\times\overline{I}).$$

If A is a set of finite perimeter, then, for a.e. $z\in\mathbb{R}^{n-1}$, A_z is of finite perimeter, and

$$\int_{\mathbb{R}^{n-1}}P(A_z)\,\mathrm{d}z\leq P(A).$$

The Plateau-type Problem

The classical Plateau problem, minimizing area among surfaces passing through a given curve, is one of the archetypical problems in Geometric Measure Theory.

Definition 2.50. Given a set $A \subset \mathbb{R}^n$, and a set E_0 of finite perimeter in $\mathbb{R}^n$, the *Plateau-type problem in A with boundary data E_0* amounts to minimizing $P(E)$ among those sets of finite perimeter E that coincide with E_0 outside A. Precisely we consider

$$\gamma(A, E_0) = \inf\left\{P(E) : E \setminus A = E_0 \setminus A\right\}. \tag{2.3}$$

Remark 2.51. Roughly speaking, prescribing that $E \setminus A = E_0 \setminus A$ we impose $E_0 \cap \partial A$ as *boundary condition* for the admissible sets E in (2.3). At the same time, the set A, being the region where E_0 can be modified to minimize perimeter, may act as an obstacle. In general we do not expect uniqueness of minimizers for this problem.

Theorem 2.52. *If $R > 0$ and $\{E_h\}_{h\in\mathbb{N}}$ are sets of finite perimeter in $\mathbb{R}^n$, with*

$$\sup_{h\in\mathbb{N}} P(E_h) < \infty,$$
$$E_h \subset B_R, \quad \forall h \in \mathbb{N},$$

then there exists E of finite perimeter in $\mathbb{R}^n$ and $h(k) \to \infty$ as $k \to \infty$, with

$$E_{h(k)} \to E, \quad \mu_{E_{h(k)}} \xrightarrow{*} \mu_E, \quad E \subset B_R,$$

with $B_R = B(x_R, \rho_R)$.

Proposition 2.53 (Existence of minimizers for the Plateau-type problem)**.** *Let $A \subset \mathbb{R}^n$ be a bounded set and let E_0 be a set of finite perimeter in $\mathbb{R}^n$. Then there exists a set of finite perimeter E such that $E \setminus A = E_0 \setminus A$ and $P(E) \le P(F)$ for every F such that $F \setminus A = F \setminus E_0$. In particular, E is a minimizer in the variational problem* (2.3).

Proof. Since E_0 itself is admissible in Equation (2.3), we have $\gamma = \gamma(A, E_0) < \infty$. Let us now consider a minimizing sequence $\{E_h\}_{h\in\mathbb{N}}$ in Equation (2.3),

$$E_h \setminus A = E_0 \setminus A, \quad P(E_h) \le P(E_0), \quad \lim_{h\to\infty} P(E_h) = \gamma.$$

If $M_h = E_h \Delta E_0 = (E_h \setminus E_0) \cup (E_0 \setminus E_h)$, then, by Lemma 2.33, M_h is a set of finite perimeter with

$$P(M_h) \le 2P(E_h) + 2P(E_0) \le 4P(E_0).$$

Since A is bounded and $M_h \subset A$, by Theorem 2.52 there exists a set of finite perimeter M such that, up to extracting a subsequence, we have $M_h \to M$. As

$$E_h = (E_0 \cup M_h) \setminus (E_0 \cap M_h),$$

and since $M_h \to M$, we find that $E_h \to E$, where we have set

$$E = (E_0 \cup M) \setminus (E_0 \cap M).$$

In particular, $E \setminus A = E_0 \setminus A$, and, by the lower semicontinuity of perimeter,

$$\gamma \le P(E) \le \liminf_{h\to\infty} P(E_h) = \gamma. \qquad \square$$

2.6 First and Second variation of perimeter

One of the most fundamental ideas in the Calculus of Variations, is that of deriving necessary conditions for minimality from the basic rules of Calculus by looking at curves of competitors which *pass through* a given candidate minimizer. Let us examine the case of a perimeter minimizer A into some open set Ω. We construct a curve of competitors *passing through* A by fixing a compactly supported smooth vector field $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$, and noticing that, for small values of a real parameter t, the maps

$$u_t : \mathbb{R}^n \to \mathbb{R}^n, \quad u_t(x) = x + t\varphi(x), \quad x \in \mathbb{R}^n,$$

define a one-parameter family of diffeomorphism of $\mathbb{R}^n$, with $u_0(x) = x$ and $u_t = u_0$ outside the support φ. Therefore we have $A\Delta u_t(A) \subset\subset \Omega$ whenever t is small enough, and we may infer (up to differentiability issues) the necessary conditions to the perimeter minimality of A in Ω,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} P(u_t(A),\Omega) = 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} P(u_t(A),\Omega) \ge 0.$$

First variation of perimeter

Definition 2.54. We say that $u : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$ if u is smooth, bijective, and has a smooth inverse $g = u^{-1}$. If A is an open set with C^1-boundary, then $u(A)$ is still an open set with C^1-boundary. So, from $u\{\psi = 0\} = \{\psi \circ g = 0\}$ and $\nabla(\psi \circ g) = (\nabla g)^*[(\nabla \psi \circ g)]$ we find

$$\nu_{u(A)}(y) = \frac{\nabla g(y)^* \nu_A(g(y))}{|\nabla g(y)^* \nu_A(g(y))|}, \quad \forall y \in \partial u(A) = u(\partial A).$$

Proposition 2.55 (Diffeomorphic images of sets of finite perimeter)**.** *If A is a set of locally finite perimeter in $\mathbb{R}^n$ and u is a diffeomorphism of $\mathbb{R}^n$ with $g = u^{-1}$, then $u(A)$ is a set of locally finite perimeter in $\mathbb{R}^n$ with*

$$\mathcal{H}^{n-1}(u(\partial^* A)\Delta\partial^* u(A)) = 0,$$

$$\int_{\partial^* u(A)} \varphi \nu_{u(A)} \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial^* A} (\varphi \circ u) J u (\nabla g \circ u)^* \nu_A \,\mathrm{d}\mathcal{H}^{n-1},$$

for every $\varphi \in C_c^0(\mathbb{R}^n)$. In particular, for every Borel set $\Omega \subset \mathbb{R}^n$,

$$\mathcal{H}^{n-1}(\Omega \cap \partial^* u(A)) = \int_{g(\Omega)\cap\partial^* A} Ju \big|(\nabla g \circ u)^* \nu_A\big| \,\mathrm{d}\mathcal{H}^{n-1}.$$

Definition 2.56. A *one parameter family of diffeomorphism* of $\mathbb{R}^n$ is a smooth function

$$(x,t)\in\mathbb{R}^n\times(-\varepsilon,\varepsilon)\longmapsto u(t,x)=u_t(x)\in\mathbb{R}^n,\quad \varepsilon>0,$$

such that, for each fixed $|t|<\varepsilon$, $u_t:\mathbb{R}^n\to\mathbb{R}$ is a diffeomorphism of $\mathbb{R}^n$. Given an open set Ω in $\mathbb{R}^n$, $\{u_t\}_{|t|<\varepsilon}$ is a *local variation in* Ω if it defines a one-parameter family of diffeomorphism such that

$$\begin{aligned} u_0(x)&=x, && \forall x\in\mathbb{R}^n,\\ \{x\in\mathbb{R}^n: u_t(x)\neq x\}&\subset\subset A, && \forall |t|<\varepsilon. \end{aligned}$$

Remark 2.57. It is easily seen that, if $\{u_t\}_{|t|<\varepsilon}$ is a local variation in Ω, then $u_t(A)\Delta\Omega\subset\subset\Omega$, for all $A\subset\mathbb{R}^n$, and the following Taylor's expansions holds uniformly on $\mathbb{R}^n$,

$$u_t(x)=x+t\varphi(x)+\mathscr{O}(t^2),\quad \nabla u_t(x)=\mathrm{Id}+t\,\nabla\varphi(x)+\mathscr{O}(t^2),$$

where $\varphi\in C_c^\infty(\Omega;\mathbb{R}^n)$ is the *initial velocity* of $\{u_t\}_{|t|<\varepsilon}$,

$$\varphi(x)=\frac{\partial u_t}{\partial t}(x,0),\quad x\in\mathbb{R}^n.$$

Conversely, starting from $\varphi\in C_c^\infty(\Omega;\mathbb{R}^n)$ there are two general ways to construct a local variation $\{u_t\}_{|t|<\varepsilon}$ in Ω having φ as initial velocity. The first naive method, consist of setting

$$u_t(x)=x+t\varphi(x),\quad x\in\mathbb{R}^n.$$

The second method relies on standard ODE theory, and consists of solving the Cauchy problem (parametrized with respect to the initial condition $x\in\mathbb{R}^n$)

$$\frac{\partial}{\partial t}u(t,x)=\varphi(u(t,x))\quad\text{and}\quad u(0,x)=x,\quad x\in\mathbb{R}^n,$$

for small values of t. In both cases, we say that $\{u_t\}_{|t|<\varepsilon}$ is a *local variation associated with* φ.

Theorem 2.58 (First variation of perimeter)**.** *We aim to compute the* first variation of perimeter *(relative to Ω) with respect to local variations $\{u_t\}_{|t|<\varepsilon}$:*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} P(u_t(A);\Omega),\quad \forall\varphi\in C_c^\infty(\Omega;\mathbb{R}^n)\ \textit{given.}$$

If Ω is an open set in $\mathbb{R}^n$, A is a set of locally finite perimeter, and $\{u_t\}_{|t|<\varepsilon}$ is a local variation in Ω, then

$$P(u_t(A);\Omega)=P(A;\Omega)+t\int_{\partial^*A}\mathrm{div}_A\,\varphi\,\mathrm{d}\mathscr{H}^{n-1}+\mathscr{O}(t^2),$$

where φ is the initial velocity of $\{u_t\}_{|t|<\varepsilon}$ and $\mathrm{div}_A\,\varphi:\partial^*A\to\mathbb{R}$,

$$\mathrm{div}_A\,\varphi(x)=\mathrm{div}\,\varphi(x)-\nu_A\cdot\nabla\varphi(x)\nu_A(x),\quad x\in\partial^*A,$$

is a Borel function called boundary divergence of φ on A.

Remark 2.59. If A is an open set with C^2-boundary, then by applying the Gauss-Green Theorem 2.9 for Surfaces to $M = \partial A$ we find that

$$\int_{\partial A} \operatorname{div}^{\partial A} \varphi \, d\mathscr{H}^{n-1} = \int_{\partial A} \varphi \cdot \mathbf{H}_{\partial A} \, d\mathscr{H}^{n-1}, \quad \forall \varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n),$$

where $\operatorname{div}^{\partial A}$ denotes the tangential divergence of φ with respect to ∂A, and where $\mathbf{H}_{\partial A} = H_{\partial A} \nu_A$ is the mean curvature vector to ∂A. Remembering that $\operatorname{div}_A T = \operatorname{div}^{\partial A} T$, similarly, we shall set $\mathbf{H}_A = \mathbf{H}_{\partial A}$ and $H_A = H_{\partial A}$; then the first variation of perimeter on open sets with C^2-boundary takes the form

$$\frac{d}{dt}\bigg|_{t=0} P(u_t(A); \Omega) = \int_{\partial A} (T \cdot \nu_A) H_A \, d\mathscr{H}^{n-1},$$

and if A is of locally finite perimeter, then the *distributional mean curvature vector* of A in Ω open is the functional $\mathbf{H}_A : C^\infty_c(A; \mathbb{R}^n) \to \mathbb{R}$, defined by the formula

$$\langle \mathbf{H}_A, \varphi \rangle = \int_{\partial^* A} \operatorname{div}_A \varphi \, d\mathscr{H}^{n-1},$$

and $\mathbf{H}_A = \mathbf{H}_{\mathbb{R}^n \setminus A}$. Moreover, if $\Omega \cap \partial A$ is a C^2-hypersurface in $\mathbb{R}^n$, then $\mathbf{H}_A$ defines a signed Radon measure on Ω, with $\mathbf{H}_A = H_A \nu_A \mathscr{H}^{n-1} \llcorner (\Omega \cap \partial A)$.

Remark 2.60. A set of locally finite perimeter A is *stationary for perimeter* in an open set Ω if

$$\operatorname{supp} \mu_A = \partial A \quad \text{and} \quad \frac{d}{dt}\bigg|_{t=0} P(u_t(A); \Omega) = 0,$$

whenever $\{u_t\}_{|t|<\varepsilon}$ is a local variation in Ω.

Corollary 2.61 (Vanishing mean curvature)**.** *A set of locally finite perimeter A is stationary for perimeter in the open set Ω if and only if*

$$\int_{\partial^* A} \operatorname{div}_A \varphi \, d\mathscr{H}^{n-1} = 0, \quad \forall \varphi \in C^1_c(\Omega; \mathbb{R}^n).$$

In particular, A has vanishing distributional mean curvature in Ω.

Second variation of perimeter

Remark 2.62. If A is an open set with C^2-boundary in Ω, then there exists an open set Ω' with $(\Omega \cap \partial A) \subset \Omega' \subset \Omega$ such that the *signed distance function* $s_A : \mathbb{R}^n \to \mathbb{R}$ of A,

$$s_A(x) = \begin{cases} \operatorname{dist}(x, \partial A), & \text{if } x \in \mathbb{R}^n \setminus A, \\ -\operatorname{dist}(x, \partial A), & \text{if } x \in A, \end{cases}$$

satisfies $s_A \in C^2(\Omega')$. We may thus define a vector field $\psi_A \in C^1(\Omega'; \mathbb{R}^n)$ and a tensor field $\Omega_A \in C^0(\Omega'; \mathbf{Sym}(\mathrm{n}))$ by setting

$$\psi_A = \nabla s_A, \quad \Omega_A = \nabla^2 s_A \quad \text{on } \Omega'.$$

It turns out that ψ_A is an extension to Ω' of the outer unit normal ν_A to A, with the property that $|\psi_A| = 1$ on Ω'. Moreover, on $B(x,\rho)$ we have

$$\Omega_A \psi_A = 0 \quad \text{and} \quad \psi_A \cdot (\Omega_A e) = 0, \quad \forall e \in \mathbb{R}^n,$$

so, in geometric terms, if $y \in B(x,\rho) \cap \partial A$, then $\Omega_A(y)$, seen as a symmetric tensor on $T_y \partial A \otimes T_y \partial A$, is the *second fundamental form of* ∂A at y.

Theorem 2.63 (Second variation of perimeter)**.** *If A is an open set with C^2-boundary in the open set Ω, $\zeta \in C_c^\infty(\Omega)$ and $\{u_t\}_{|t|<\varepsilon}$ is a local variation associated with the normal vector field $\varphi = \zeta \psi_A \in C_c^1(\Omega; \mathbb{R}^n)$, then*

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} P(u_t(A); \Omega) = \int_{\partial A} |\nabla_A \zeta|^2 + \left(H_A^2 - |\Omega_A|^2\right) \zeta^2 \, \mathrm{d}\mathscr{H}^{n-1},$$

where $\nabla_A \zeta = \nabla \zeta - (\nu_A \cdot \nabla \zeta)\nu_A$ denotes the tangential gradient of ζ with respect to the boundary of A. In particular, if A is a perimeter minimizer in Ω, then

$$\int_{\partial A} |\nabla_A \zeta|^2 - |\Omega_A|^2 \zeta^2 \, \mathrm{d}\mathscr{H}^{n-1} \geq 0, \quad \textit{for every } \zeta \in C_c^\infty(\Omega).$$

2.7 Coarea formula

As Didone for the isoperimetric problem in Section 2.5, we approach the sets by cutting their area.

If $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then, by the Morse-Sard Lemma (see Lemma 2.66), the set

$$\{u = t\} = \{x \in \mathbb{R}^n : u(x) = t\}$$

is the smooth hypersurface in $\mathbb{R}^n$ for a.e. $t \in \mathbb{R}$. It is often natural to look at the integral over $t \in \mathbb{R}$ of the $\mathscr{H}^{n-1}$-dimensional measure of the slices $A \cap \{u = t\}$ of a Borel set $A \subset \mathbb{R}^n$,

$$\int_{\mathbb{R}} \mathscr{H}^{n-1}(A \cap \{u = t\}) \, \mathrm{d}t,$$

which, by the *coarea formula*, coincides with the *total variation* of u over A,

$$\int_A |\nabla u| = \int_{\mathbb{R}} \mathscr{H}^{n-1}(A \cap \{u = t\}) \, \mathrm{d}t.$$

Theorem 2.64 (Coarea formula)**.** *If $u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function and $A \subset \mathbb{R}^n$ is open, then $t \in \mathbb{R} \longmapsto P(\{u > t\}; A)$ is a Borel function on $\mathbb{R}^n$ with*

$$\int_A |\nabla u| = \int_{\mathbb{R}} P(\{u > t\}; A) \, \mathrm{d}t \tag{2.4}$$

as elements of $[0, \infty]$. In other words, the total variation of a function is also the accumulated surfaces of all its level sets.

Proof. In the proof, we are going to use the following *layer-cake formula*. If $u \in \mathrm{L}^1(\mathbb{R}^n)$, $u \geq 0$, and $v \in \mathrm{L}^\infty(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} u(x)v(x)\,\mathrm{d}x = \int_0^\infty \mathrm{d}t \int_{\{u>t\}} v(x)\,\mathrm{d}x. \tag{2.5}$$

Indeed, for every $x \in \mathbb{R}^n$,

$$u(x) = \int_{\mathbb{R}} \chi_{(0,u(x))}(t)\,\mathrm{d}t = \int_{\mathbb{R}} \chi_{(0,\infty)}(t)\chi_{\{u>t\}}(x) = \int_0^\infty \chi_{\{u>t\}}(x)\,\mathrm{d}t,$$

and thus, by Fubini's Theorem,

$$\int_{\{u\geq 0\}} u(x)v(x)\,\mathrm{d}x = \int_{\{u\geq 0\}} v(x) \int_0^\infty \chi_{\{u>t\}}(x)\,\mathrm{d}t = \int_0^\infty \mathrm{d}t \int_{\{u\geq t\}} v(x)\,\mathrm{d}x.$$

- *Step one*: If $T \in C_c^1(A;\mathbb{R}^n)$, then $\int_{\{u>t\}} \operatorname{div} T$ is a Borel measurable function of $t \in \mathbb{R}$. Indeed, it is the difference of two increasing functions of $t \in \mathbb{R}$, namely

  $$\int_{\{u>t\}} \operatorname{div}\varphi = \int_{\{u>t\}} (\operatorname{div} T)^+ - \int_{\{u>t\}} (\operatorname{div} T)^-.$$

  If $\mathscr{F}$ is countable and dense in $C_c^\infty(A;\mathbb{R}^n)$, then, by Definition 2.22,

  $$P(\{u>t;A\}) = \sup\left\{\int_{\{u>t\}} \operatorname{div} T : T \in \mathscr{F}, \sup_{\mathbb{R}^n}|T| \leq 1\right\}.$$

  Since the supremum of countably many Borel functions is a Borel function, we have proved that $t \in \mathbb{R} \longmapsto P(\{u>t\};A)$ is a Borel function.

- *Step two*: We prove that if u is a non-negative Lipschitz function then

  $$\int_A |\nabla u| \leq \int_0^\infty P(\{u>t\};A)\,\mathrm{d}t,$$

  for every open set $A \in \mathbb{R}^n$ (in particular, if the left-hand side is infinite then the right-hand side is infinite too). If $T \in C_c^\infty(A;\mathbb{R}^n)$, $|T| \leq 1$, then by Definition 2.22

  $$\int_{\{u>t\}} \operatorname{div} T \leq P(\{u>t\};A), \quad t>0. \tag{2.6}$$

  By the distributional divergence Theorem and by Equation (2.5) (with $v = \operatorname{div} T$),

  $$-\int_A \nabla u \cdot T = \int_{\mathbb{R}^n} u \operatorname{div} T = \int_0^\infty \mathrm{d}t \int_{\{u>t\}} \operatorname{div} T \leq \int_0^\infty P(\{u>t\};A)\,\mathrm{d}t. \tag{2.7}$$

  Let K be a compact subset of A and define $S : \mathbb{R}^n \to \mathbb{R}^n$ as

  $$S(x) = -\chi_{K\cap\{\nabla u \neq 0\}}(x)\frac{\nabla u(x)}{|\nabla u(x)|}, \quad x \in \mathbb{R}^n,$$

so that S is bounded Borel measurable vector field with $|S| \leq 1$. For every $\varepsilon < \mathrm{dist}(K, \partial A)$ we have that the convolution $S_\varepsilon = (S * \rho_\epsilon) \in C_c^\infty(A; \mathbb{R}^n)$ with $|S_\varepsilon| \leq 1$ and $S_\varepsilon(x) \to S(x)$ for a.e. $x \in \mathbb{R}^n$. We let $\varphi = S_\varepsilon$ and $\varepsilon \to 0$ in Equation (2.7) to find that

$$\int_K |\nabla u| \leq \int_0^\infty P(\{u > t\}; A)\, \mathrm{d}t,$$

(where the left-hand side is finite). Since K is arbitrary, we find Equation (2.6).

- *Step three*: We prove that if u is non-negative Lipschitz function then

$$\int_A |\nabla u| \geq \int_0^\infty P(\{u > t\}; A)\, \mathrm{d}t.$$

To this end we consider the increasing function $m : \mathbb{R} \to [0, \infty)$ defined as

$$m(t) = \int_{A \cap \{u \leq t\}} |\nabla u|, \quad t \in \mathbb{R}.$$

The classical derivative $m'(t)$ exists for a.e. $t \in \mathbb{R}$, and

$$\int_0^\infty m'(t)\, \mathrm{d}t \leq \lim_{t \to \infty} m(t) - \lim_{t \to -\infty} m(t) = \int_A |\nabla u|.$$

We are thus left to show that

$$m'(t) \geq P(\{u > t\}; A), \quad \text{for a.e. } t \geq 0. \tag{2.8}$$

Given $t \geq 0$ and $\varepsilon > 0$, define a piecewise affine function $\psi : [0, \infty) \to [0, 1]$ as

$$\psi(s) = \begin{cases} 1, & s \in [t + \varepsilon, \infty), \\ \varepsilon^{-1}(s - t), & s \in [t, t + \varepsilon), \\ 0, & s \in [0, t). \end{cases}$$

From the chain rule, $\psi \circ u$ admits $(\psi \circ u) \nabla u = -\varepsilon^{-1} \chi_{(t, t+\varepsilon)}(u) \nabla u$ as its weak gradient on $\mathbb{R}^n$. If $T \in C_c^\infty(A; \mathbb{R}^n)$ with $|T| \leq 1$, then

$$\int_A (\psi \circ u) \operatorname{div} \varphi = -\frac{1}{\varepsilon} \int_{A \cap \{t + \varepsilon > u > t\}} \nabla u \cdot T \leq \frac{1}{\varepsilon} \int_{A \cap \{t + \varepsilon > u > t\}} |\nabla u| \leq \frac{m(t + \varepsilon - m(t)}{\varepsilon}.$$

As $\varepsilon \to 0^+$ we find that, for a.e. $t > 0$,

$$\int_{A \cap \{u > t\}} \operatorname{div} T \leq m'(t),$$

which implies Equation (2.8) by the arbitrariness of T.

- *Step four*: Finally, let $u:\mathbb{R}^n\to\mathbb{R}$ be a Lipschitz function, and consider its positive and negative parts u^+ and u^-. By the chain rule, $\nabla u^+=\chi_{\{u>0\}}\nabla u$ and $\nabla u^-=-\chi_{\{u<0\}}\nabla u$, so, from the previous steps, the Coarea formula (2.4) holds true for u^+ and u^-. Hence,

$$\begin{aligned}\int_A|\nabla u|=\int_A|\nabla u^+|+\int_A|\nabla u^-|&=\int_0^\infty P(\{u^+>t\};A)\,\mathrm{d}t+\int_0^\infty P(\{u^->t\};A)\,\mathrm{d}t\\ &=\int_0^\infty P(\{u>t\};A)\,\mathrm{d}t+\int_0^\infty P(\{u<-t\};A)\,\mathrm{d}t\\ &=\int_0^\infty P(\{u>t\};A)\,\mathrm{d}t+\int_{-\infty}^0 P(\{u<t\};A)\,\mathrm{d}t.\end{aligned}\tag{2.9}$$

Moreover, by the complement property in Property 2.26, we have that

$$P(\{u<t\};A)=P(\{u\ge t\};A).$$

Since $|\{u=t\}|=0$ for a.e. $t\in\mathbb{R}$, we thus have $P(\{u\ge t\};A)=P(\{u>t\};A)$ for a.e. $t\in\mathbb{R}$. Hence the Coarea formula follows from Equation (2.9). □

Remark 2.65. A special case of Coarea formula (2.4) is Fubini's Theorem 1.22. Another special case is integration in spherical coordinates in which the integral of a function on $\mathbb{R}^n$ is related to the integral of the function over spherical shells (level sets of the radial function).

Lemma 2.66 (Morse-Sard)**.** *If $u\in C^\infty(\mathbb{R}^n)$ and $A=\{x\in\mathbb{R}^n:\nabla u(x)=0\}$, then $|u(A)|=0$. In particular, $\{u=t\}=\{x\in\mathbb{R}^n:u(x)=t\}$ is a smooth hypersurface in $\mathbb{R}^n$ for a.e. $t\in\mathbb{R}$.*

Remark 2.67. By the Morse-Sard Lemma, if $u\in C^\infty(\mathbb{R}^n)$, then, for a.e. $t\in\mathbb{R}$, $\{u>t\}$ is an open set with smooth boundary. Hence $P(\{u>t\};\Omega)=\mathscr{H}^{n-1}(\Omega\cap\{u=t\})$ for a.e. $t\in\mathbb{R}$ and for every $A\subset\mathbb{R}^n$ open. So, for every Borel set $A\subset\mathbb{R}^n$,

$$\int_A|\nabla u|=\int_{\mathbb{R}^n}\mathscr{H}^{n-1}(A\cap\{u=t\})\,\mathrm{d}t.$$

Example 2.68. Let u be the non-negative Lipschitz function of Figure 6, with compact support on $\mathbb{R}$. By the Fundamental Theorem of Calculus

$$\int_{a_0}^{a_1}|u'|=\int_{a_3}^{a_4}|u'|=t_2\quad\text{with}\quad\int_{a_1}^{a_3}|u'|=\int_{a_2}^{a_3}|u'|=t_2-t_1\implies\int_{\mathbb{R}}|u'|=4t_2-2t_1.$$

At the same time

$$P(\{u>t\})=\begin{cases}0 & \text{if } t\in(-\infty,0)\cup(t_2,+\infty)\\ 2 & \text{if } t\in(0,t_1)\\ 4 & \text{if } t\in(t_1,t_2),\end{cases}$$

so that, as expected,

$$\int_{\mathbb{R}}P(\{u>t\})\,\mathrm{d}t=2t_1+4(t_2-t_1).$$

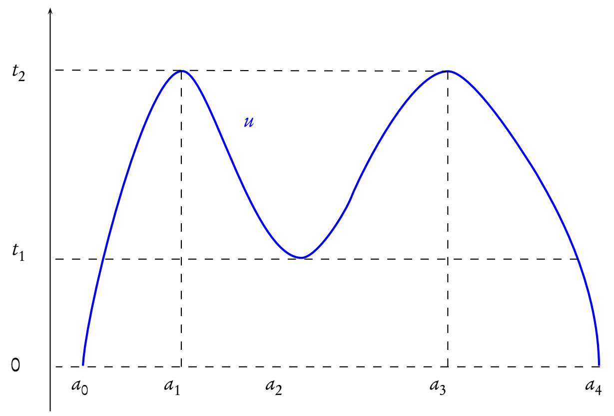


Figure 6: Graphic interpretation of coarea formula for Example 2.68.

Theorem 2.69 (Approximation by smooth sets)**.** *A Lebesgue measurable set $A \subset \mathbb{R}^n$ is of locally finite perimeter if and only if there exists a sequence $\{A_h\}_{h\in\mathbb{N}}$ of open sets with smooth boundary in $\mathbb{R}^n$ and $\varepsilon_h \to 0^+$, such that*

$$A_h \xrightarrow{loc} A, \qquad \sup_{h\in\mathbb{N}} P(A_h; B_R) < \infty, \quad \forall R > 0$$

$$|\mu_{A_h}| \xrightarrow{*} |\mu_A| \qquad \partial A_h \subset \chi_{\varepsilon_h}(\partial A).$$

In particular, $P(A_h;\Omega) \to P(A;\Omega)$ whenever $P(A;\partial\Omega) = 0$. Moreover, if $|A| < \infty$, then $A_h \to A$ while if $P(A) < \infty$, then $P(A_h) \to P(A)$.

Remark 2.70 (Approximation by polyhedra)**.** When A is a set of finite perimeter in $\mathbb{R}^n$ with $|A| < \infty$, we may also approximate A by a sequence of open bounded sets with polyhedral boundary.

Theorem 2.71 (Coarea formula revised)**.** *If $u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then*

$$\int_A |\nabla u| = \int_{\mathbb{R}} \mathscr{H}^{n-1}(A \cap \{u = t\})\,\mathrm{d}t \quad \textit{for every Borel set } A \subset \mathbb{R}^n. \tag{2.10}$$

Remark 2.72. If A is of finite perimeter and $u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then for a.e. $t \in \mathbb{R}$,

$$P(A \cap \{u > t\}) = P(A; \{u > t\}) + \mathscr{H}^{n-1}(A \cap \{u = t\}).$$

Lemma 2.73 (Slicing the set of critical points)**.** *If $u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function and $A = \{x \in \mathbb{R}^n : \nabla u = 0\}$, then for a.e. $t \in \mathbb{R}$,*

$$\mathscr{H}^{n-1}(A \cap \{u = t\}) = 0,$$

as, close to a point x where $\nabla u(x) = 0$, u is almost constant, so $u(\{\nabla u(x) = 0\})$ is expected to have zero Lebesgue measure.

Example 2.74. From the coarea formula, we immediately deduce that if $u : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, then, for a.e. $t > 0$, the open set $\{u > t\}$ is of locally finite perimeter in $\mathbb{R}^n$.

Moreover, if $\int_{\{u>s\}} |\nabla u| < \infty$ for some $s \in \mathbb{R}$, then for a.e. $t > s$ the open set $\{u > t\}$ is of locally finite perimeter in $\mathbb{R}^n$. Hence, by Theorem 2.71 and De Giorgi's structure Theorem 2.43, for every $A \subset \mathbb{R}^n$ open,

$$\int_A |\nabla u| = \int_{\mathbb{R}} \mathscr{H}^{n-1}(A \cap \partial^*\{u > t\})\,\mathrm{d}t. \tag{2.11}$$

Remark 2.75. Equation (2.10) is stronger than Equation (2.11) because $\{u = t\}$ is the topological boundary of $\{u > t\}$. The $\mathscr{H}^{n-1}$-equivalence of the topological boundary $\partial\{u > t\}$ and the reduced boundary $\partial^*\{u > t\}$ of a.e. super-level set of a Lipschitz function is a corollary of Theorem 2.71.

Remark 2.76. If A is of finite perimeter and $u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then for a.e. $t \in \mathbb{R}$,

$$P(A \cap \{u > t\}) = P(A; \{u > t\}) + \mathscr{H}^{n-1}(A \cap \{u = t\}).$$

CHAPTER 3

Functions of Bounded Variation

Functions of *Bounded Variation* were singled out as those functions for which a control on the oscillations is possible, suitable to ensure the convergence of the Fourier series. These functions in $\mathbb{R}$ have been introduced by C. Jordan in 1881 in connexion with Dirichlet's test for the convergence of Fourier series. In the following we adopt the notation of [Ambrosio *et al.*(2000)] where ν_A is the inner normal of set A.

Theorem 3.1. *Let $u : \mathbb{R} \to \mathbb{R}$ be a 2π periodic summable function.*

- *If u has bounded variation in an open interval I then its Fourier series converges to*

$$\frac{1}{2}\big(u(x^+) + u(x^-)\big), \quad \forall x \in I.$$

- *If in addition u is continuous in I then its Fourier series converges uniformly to u on every closed interval $J \subset I$.*

Jordan also pointed out the canonical decomposition of a BV function as the difference of two increasing functions. In 1905, G. Vitali gave the first definition of a BV function of two real variables [Vitali(1905)].

Definition 3.2. Given $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, for every rectangle $R \subset \Omega$ with vertices $P_{i,j} = (x_i, y_j)$ with $x_1 < x_2$, $y_1 < y_2$, setting

$$\mathrm{dV}(u, R) = u(p_{1,1}) - u(P_{1,2}) - u(P_{2,1}) + u(P_{2,2}),$$

we define the *double variation* of u in Ω as

$$\mathrm{dV}(u, \Omega) = \sup\left\{\sum_{i=1}^{n} \mathrm{dV}(u, R_i) : R_i \subset \Omega \text{ pairwise disjoint rectangles}\right\};$$

then u is said to have bounded variation in Ω if $\mathrm{dV}(u, \Omega)$ is finite.

L. Tonelli noticed that the double variation was not the right generalization of the one-dimensional variation, because it contains second order elements, related to the *curvature* of the graph of u rather than its area. Thus he proposed to call functions of bounded variation in $\Omega \subset \mathbb{R}^2$ those continuous functions for which the *surface area* of the projections of the graph of u onto the vertical coordinate planes (counting multiplicities) are finite. Taking for simplicity $\Omega = (0,1)^2$, this amounts to require

$$\int_0^1 \mathrm{pV}(u_x,(0,1))\,\mathrm{d}x < \infty \quad \text{and} \quad \int_0^1 \mathrm{pV}(u_y,(0,1))\,\mathrm{d}y < \infty,$$

where $u_x(\,\cdot\,) = u(x,\,\cdot\,)$, $u_y(\,\cdot\,) = u(\,\cdot\,,y)$ and pV defined as in Equation (3.8). However this approach depends on the choice of the coordinate axes if u is not continuous.

It was only with G. Fichera and E. De Giorgi that the theory of BV functions was tied with distributions. Fichera considered the set functions $T_i u$, defined, for Q cube in Ω with sides parallel to the coordinate axes, by the following equality:

$$T_i u(Q) = \int_{\partial Q} u \nu_i \,\mathrm{d}\mathscr{L}^{n-1}, \quad i = 1,\dots,n,$$

where ν_i is the i-th component of the outward pointing unit normal to ∂Q. Then u is BV if the set functions $T_i u$ have finite total variation: the fact that u is in BV means that its partial derivatives, in the sense of distributions, are measures with finite total variation. De Giorgi proved that given $u \in \mathrm{L}^\infty(\mathbb{R}^n)$ and set, for $\lambda > 0$, $\varphi_\lambda(x) = (\pi\lambda)^{n/2}\exp\{-|x|^2/\lambda\}$, the following limit exists

$$I(u) = \lim_{\lambda \to 0} \int_{\mathbb{R}^n} |\nabla(u * \varphi_\lambda)|\,\mathrm{d}x,$$

and that $I(u)$ is finite if and only if the distributional gradient of u is an $\mathbb{R}^n$-valued measure Du with finite total variation. Moreover, if this is the case, $I(u) = |Du|(\mathbb{R}^n)$. If u is the characteristic function of a measurable set A, then De Giorgi proved that $I(u)$ is the perimeter of A in the sense of Caccioppoli, i.e.

$$P(A) = \inf\left\{ \liminf_{h \to \infty} P(A_h) : \partial A_h \text{polyhedral}, \lim_{h \to \infty} \mathscr{L}^n(A_h \Delta A) = 0 \right\},$$

where the perimeter of the approximating polyhedra A_h is defined in an elementary way.

Finally, in 1964, M. Miranda introduced the quantity $V(u,\Omega)$ of Equation (3.3) in order to characterise the functions $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ whose distributional gradient is a measure [Miranda(1964)].

3.1 Functions of Bounded Variation

The standard definition of *Bounded Variation* involve the distributive derivatives and Radon measure but the true link between measure theory and functional analysis is inside Riesz Theorem.

Definition 3.3 (Function of Bounded Variation)**.** Let $u \in \mathrm{L}^1(\Omega)$ and Ω a generic open set in $\mathbb{R}^n$: we say that u is a *function of Bounded Variation in* Ω if the distributional derivative of u is representable by a finite Radon measure in Ω, i.e. if

$$\int_\Omega u \nabla \varphi = -\int_\Omega \varphi \, \mathrm{d}Du, \quad \forall \varphi \in C_c^\infty(\Omega),$$

or, in components,

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = -\int_\Omega \varphi \, \mathrm{d}D_i u, \quad \forall \varphi \in C_c^\infty(\Omega), \quad i = 1, \dots, n, \tag{3.1}$$

for some $\mathbb{R}^n$-valued measure $Du = (D_1 u, \dots, D_n u)$ in Ω. The vector space of all functions of Bounded Variation in Ω is denoted by $\mathrm{BV}(\Omega)$.

Remark 3.4. A smoothing argument shows that the integration by parts in Equation (3.1) is still true for any $\varphi \in C_c^1(\Omega)$, or even for Lipschitz functions φ with compact support in Ω. So Equation (3.1) can be summarized into a single one by writing

$$\int_\Omega u \operatorname{div} \varphi \, \mathrm{d}x = -\sum_{i=1}^n \int_\Omega \varphi_i \, \mathrm{d}D_i u, \quad \forall \varphi \in [C_c^1(\Omega)]^n.$$

We will use the same notation also for functions $u \in [\mathrm{BV}(\Omega)]^m$: in this case, Du is a $m \times n$ matrix of measures $D_i u^\alpha$ in Ω satisfying:

$$\int_\Omega u^\alpha \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = -\int_\Omega \varphi \, \mathrm{d}D_i u^\alpha, \quad \forall \varphi \in C_c^1(\Omega), \, i = 1, \dots, n; \, \alpha = 1, \dots, m,$$

or equivalently,

$$\sum_{\alpha=1}^m \int_\Omega u^\alpha \operatorname{div} \varphi^\alpha \, \mathrm{d}x = -\sum_{\alpha=1}^m \sum_{i=1}^n \int_\Omega \varphi_i^\alpha \, \mathrm{d}D_i u^\alpha, \quad \forall \varphi \in [C_c^1(\Omega)]^{mn}. \tag{3.2}$$

Remark 3.5. The Sobolev Space $\mathrm{W}^{1,1}(\Omega)$ is contained in $\mathrm{BV}(\Omega)$: for any $u \in \mathrm{W}^{1,1}(\Omega)$ the distributional derivative is given by $\nabla u \mathscr{L}^n$. This inclusion is strict: there exist functions $u \in \mathrm{BV}(\Omega)$ such that Du is singular with respect to $\mathscr{L}^n$, for instance, the Heavyside function $\chi_{(0,\infty)}$, whose distributional derivative is the Dirac measure $\delta_0 \notin \mathrm{L}^1$, belongs to $\mathrm{BV}((0,\infty)) \setminus \mathrm{W}^{1,1}((0,\infty))$.

Proposition 3.6 (Properties of Du)**.** *Let* $u \in [\mathrm{BV}_{\mathrm{loc}}(\Omega)]^m$. *Then*

- *if* $Du = 0$, u *is (equivalent to a) constant in any connected component of* Ω*;*
- *for any locally Lipschitz function* $\psi : \Omega \to \mathbb{R}$ *the function* $u\psi$ *belongs to* $[\mathrm{BV}_{\mathrm{loc}}(\Omega)]^m$ *and*

$$D(u\psi) = \psi Du + (u \otimes \nabla \psi)\mathscr{L}^n.$$

One of the main advantages of the BV space it that it includes, unlike Sobolev spaces, characteristic functions of sufficiently regular sets and, more generally, piecewise smooth functions.

Example 3.7. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let us assume the existence of pairwise disjoint sets with piecewise C^1 boundary $\{\Omega_i\}_{1\le i\le p}$ such that

$$\bigcup_{i=1}^{p} \Omega_i \subset \Omega \subset \bigcup_{i=1}^{p} \overline{\Omega_i}.$$

If $u_i \in C(\overline{\Omega}_i)$, we can define $u : \Omega \to \mathbb{R}$ to be equal to u_i on any subdomain Ω_i, and define it arbitrarily on the remaining negligible set Σ. By applying the Gauss-Green Theorem to any domain Ω_i, for $i = 1, \dots, p$, we find

$$\int_{\Omega_i} u \operatorname{div} \varphi \, \mathrm{d}x = -\int_{\partial\Omega_i} u_i \langle \varphi, \nu_i \rangle \, \mathrm{d}\mathscr{H}^1 - \int_{\Omega_i} \langle \nabla u, \varphi \rangle \, \mathrm{d}x, \quad \forall \varphi \in \left[C^1(\overline{\Omega}_i)\right]^2$$

where ν_i is the outer unit normal to Ω_i. Adding with respect to i these identities we find that $u \in \mathrm{BV}(\Omega)$, with Du given by

$$\sum_{i=1}^{p} u_i \nu_i \mathscr{H}^1 \llcorner (\Omega \cap \partial\Omega_i) + \nabla u \mathscr{L}^2.$$

Definition 3.8 (Variation)**.** Given $u \in [\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ the *variation* of u is:

$$V(u,\Omega) = \sup\left\{ \int_{\Omega} u \operatorname{div} \varphi \, \mathrm{d}x : \varphi \in [C^1_c(\Omega)]^n, \, \|\varphi(\mathbf{x})\|_\infty \le 1 \right\}, \tag{3.3}$$

or, generally speaking,

$$V(u,\Omega) = \sup\left\{ \sum_{\alpha=1}^{m} \int_{\Omega} u^\alpha \operatorname{div} \varphi^\alpha \, \mathrm{d}x : \varphi \in [C^1_c(\Omega)]^{mn}, \, \|\varphi(\mathbf{x})\|_\infty \le 1 \right\},$$

Integration by parts proves that $V(u,\Omega) = \int_\Omega |\nabla u| \, \mathrm{d}x$ if u is continuously differentiable in Ω.

Remark 3.9. A function $u \in [\mathrm{L}^1(\Omega)]^m$ belongs to $[\mathrm{BV}(\Omega)]^m$ if and only if $V(u,\Omega) < \infty$.

Property 3.10 (Properties of the variation)**.** We summarize some useful properties.

- *Lower semicontinuity*: $u \longmapsto V(u,\Omega)$ is lower semicontinuous in the $[\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ topology because

  $$u \longmapsto \sum_{\alpha=1}^{m} \int_{\Omega} u^\alpha \operatorname{div} \varphi^\alpha \, \mathrm{d}x$$

  is continuous in the $[\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ topology for any choice of $\varphi \in [C^1_c(\Omega)]^{mn}$. So

  $$V(u,\Omega) \le \liminf_{h\to\infty} V(u_h,\Omega).$$

- *Additivity*: $V(u,A)$ is defined for any open set $A \subset \Omega$ (with φ supported in A) that

  $$\widetilde{V}(u,B) = \inf\{V(u,A) : A \supset B, A \text{ open}\}, \quad B \in \mathscr{B}(\Omega)$$

  extends $V(u,\cdot)$ to a Borel measure in Ω.

- *Locality*: the mapping $u \longmapsto V(u,A)$ is local, i.e. $V(u,A) = V(v,A)$ if $u \equiv v$ $\mathscr{L}^n$-a.e. in $A \subset \Omega$.
- *Convexity*: given two maps $u_1 \longmapsto V(u_1,\Omega)$, $u_2 \longmapsto V(u_2,\Omega)$ and $t \in [0,1]$, then

$$V(t u_1 + (1-t)u_2,\Omega) \leq t V(u_1,\Omega) + (1-t)V(u_2,\Omega).$$

- *One homogeneous*: $V(tu,\Omega) = tV(u,\Omega)$.

Remark 3.11. Since $u \longmapsto V(u,\Omega)$ is lower semicontinuous in the $[\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ topology, this provides a useful method of showing that some function $u \in [\mathrm{L}^1(\Omega)]^m$ belongs to $[\mathrm{BV}(\Omega)]^m$: one needs only to approximate $u \in [\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ by functions $\{u_h\}_{h\in\mathbb{N}}$ whose variations $V(u_h,\Omega)$ are equibounded.

Proposition 3.12 (Variation of BV(Ω) functions)**.** *Let $u \in [\mathrm{L}^1(\Omega)]^m$. Then, u belongs to $[\mathrm{BV}(\Omega)]^m$ if and only if $V(u,\Omega) < \infty$. In addition, $V(u,\Omega)$ coincides with $|Du|(\Omega)$ for any $u \in [\mathrm{BV}(\Omega)]^m$ and $u \longmapsto |Du|(\Omega)$ is lower semicontinuous in $[\mathrm{BV}(\Omega)]^m$ with respect to the $[\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ topology.*

Proof. If $u \in [\mathrm{BV}(\Omega)]^m$ we can estimate the supremum defining $V(u,\Omega)$ observing that

$$\sum_{\alpha=1}^{m} \int_\Omega u^\alpha \operatorname{div} \varphi^\alpha \,\mathrm{d}x = -\sum_{i=1}^{n}\sum_{\alpha=1}^{m} \int_\Omega \varphi_i^\alpha \,\mathrm{d}D_i u^\alpha$$

for any $\varphi \in [C_c^1(\Omega)]^{mn}$. Since in the computation of $V(u,\Omega)$ we require that $||\varphi||_\infty \leq 1$, from Proposition 1.30 we infer that

$$V(u,\Omega) \leq |Du|(\Omega) < \infty.$$

Conversely, if $V(u,\Omega) < \infty$ a homogeneity argument shows that

$$\left| \sum_{\alpha=1}^{m} \int_\Omega u^\alpha \operatorname{div} \varphi^\alpha \,\mathrm{d}x \right| \leq V(u,\Omega)||\varphi||_\infty, \quad \forall \varphi \in [C_c^1(\Omega)]^{mn}.$$

Since $[C_c^1(\Omega)]^{mn}$ is dense in $[C_0(\Omega)]^{mn}$, we can find a continuous linear functional L on $[C_0(\Omega)]^{mn}$ coinciding with

$$\varphi \longmapsto \sum_{\alpha=1}^{m} \int_\Omega u^\alpha \operatorname{div} \varphi^\alpha \,\mathrm{d}x$$

on $[C_c^1(\Omega)]^{mn}$ and satisfying $||L|| \leq V(u,\Omega)$. By Riesz Theorem, there exists a $\mathbb{R}^{mn}$-valued finite Radon measure $\mu = (\mu_i^\alpha)$ such that $||L|| = |\mu|(\Omega)$ and

$$L(\varphi) = \sum_{i=1}^{n}\sum_{\alpha=1}^{m} \int_\Omega \varphi_i^\alpha \,\mathrm{d}\mu_i^\alpha, \quad \forall \varphi \in [C_0(\Omega)]^{mn}.$$

From Equation (3.2) and the identity

$$\sum_{\alpha=1}^{m} \int_\Omega u^\alpha \operatorname{div} \varphi^\alpha \,\mathrm{d}x = \sum_{i=1}^{n}\sum_{\alpha=1}^{m} \int_\Omega \varphi_i^\alpha \,\mathrm{d}\mu_i^\alpha, \quad \forall \varphi \in [C_c^1(\Omega)]^{mn}$$

we obtain that $u \in [\mathrm{BV}(\Omega)]^m$, $Du = -\mu$ and

$$|Du|(\Omega) = |\mu|(\Omega) = ||L|| \leq V(u,\Omega).$$

Finally, the lower semicontinuity of $u \longmapsto |Du|(\Omega)$ follows directly from Property 3.10. $\square$

Remark 3.13. $|Du|(\Omega)$ will be sometimes called the *variation* of $u \in \Omega$ and this notation will be used for $\mathrm{BV}(\Omega)$ functions only.

Remark 3.14. The space $[\mathrm{BV}(\Omega)]^m$, endowed with the norm

$$||u||_{\mathrm{BV}(\Omega)} = \int_\Omega |u|\,\mathrm{d}x + |Du|(\Omega),$$

is a Banach space, but the norm-topology is too strong for many applications. Indeed, even for $m = 1$, continuously differentiable functions are not dense in $\mathrm{BV}(\Omega)$. However $[\mathrm{BV}(\Omega)]^m$ can be approximated in the $[\mathrm{L}^1(\Omega)]^m$ topology, by smooth functions whose gradients are bounded in $[\mathrm{L}^1(\Omega)]^m$. This is a refinement of the classical Meyers-Serrin Theorem for Sobolev spaces.

Theorem 3.15 (Approximation by smooth functions). *Let $u \in [\mathrm{L}^1(\Omega)]^m$. Then $u \in [\mathrm{BV}(\Omega)]^m$ if and only if there exists a sequence $\{u_h\}_{h\in\mathbb{N}}$ of functions in $[C^\infty(\Omega)]^m$ such that $u_h \to u$ in $[\mathrm{L}^1(\Omega)]^m$ and satisfying*

$$\lim_{h\to\infty} |Du_h|(\Omega) = \lim_{h\to\infty} \int_\Omega |\nabla u_h|\,\mathrm{d}x \to \int_\Omega |\nabla u|\,\mathrm{d}x = |Du|(\Omega) < \infty.$$

Since $|\cdot|_1$ is not a strictly convex norm, problems arise in many proofs where a smoothing argument, involving the Reshetnyak continuity Theorem, is used: for this reason the results stated here involve vector BV functions, since not all of them can be directly deduced from the corresponding scalar ones.

Theorem 3.16 (Reshetnyak, lower semicontinuity). *Let Ω be an open subset of $\mathbb{R}^n$ and μ, μ_h be $\mathbb{R}^m$-valued finite Radon measures in Ω; if $\mu_h \to \mu$ weakly-$*$ in Ω, then*

$$\int_\Omega u\left(x, \frac{\mu}{|\mu|}(x)\right)\mathrm{d}|\mu|(x) \le \liminf_{h\to\infty} \int_\Omega u\left(x, \frac{\mu_h}{|\mu_h|}(x)\right)\mathrm{d}|\mu_h|(x)$$

for every lower semicontinuous function $u : \Omega \times \mathbb{R}^m \to [0, \infty]$, positively 1-homogeneous and convex in the second variable.

Theorem 3.17 (Reshetnyak, continuity). *Let Ω, μ_h, μ as in Theorem 3.16; if $|\mu_h|(\Omega) \to |\mu|(\Omega)$ then*

$$\lim_{h\to\infty} \int_\Omega u\left(x, \frac{\mu_h}{|\mu_h|}(x)\right)\mathrm{d}|\mu_h|(x) = \int_\Omega u\left(x, \frac{\mu}{|\mu|}(x)\right)\mathrm{d}|\mu|(x),$$

for every continuous and bounded function $u : \Omega \times \mathrm{S}^{m-1} \to \mathbb{R}$.

As consequence of Theorem 3.15, $\mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ is an algebra: if u^i $(i = 1, 2)$ belong to this space, a simple truncation argument in conjunction with Theorem 3.15 shows that we can find sequences $\{u_h^i\}_{h\in\mathbb{N}}$ of smooth functions such that $\{|Du_h^i|(\Omega)\}_{h\in\mathbb{N}}$ converges to $|Du^i|(\Omega)$ and $\{||u_h^i||_\infty\}_{h\in\mathbb{N}}$ converges to $||u^i||_\infty$ for $i = 1, 2$. Since the functions $v_h = u_h^1 u_h^2$ converge to $v = u^1 u^2$, passing to the limit as $h \to \infty$ in the inequality

$$\int_\Omega |\nabla v_h|\,\mathrm{d}x \le ||u_h^1||_\infty \int_\Omega |\nabla u_h^2|\,\mathrm{d}x + ||u_h^2||_\infty \int_\Omega |\nabla u_h^1|\,\mathrm{d}x$$

we obtain that v belongs to $\mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and

$$|Dv|(\Omega) \le ||u^1||_\infty |Du^2|(\Omega) + ||u^2||_\infty |Du^1|(\Omega).$$

3.2 Convergence in BV space

Definition 3.18 (Weak-$*$ convergence). Let $u, u_h \in [\mathrm{BV}(\Omega)]^m$. We say that $\{u_h\}_{h\in\mathbb{N}}$ weakly-$*$ converges in $[\mathrm{BV}(\Omega)]^m$ to u if $\{u_h\}_{h\in\mathbb{N}}$ converges to u in $[\mathrm{L}^1(\Omega)]^m$ and $\{Du_h\}_{h\in\mathbb{N}}$ weakly-$*$ converges to Du in Ω, i.e.

$$\lim_{h\to\infty}\int_\Omega \varphi \,\mathrm{d}Du_h = \int_\Omega \varphi\,\mathrm{d}Du, \quad \forall\varphi\in C_0(\Omega). \tag{3.4}$$

Remark 3.19. We dont't use the weak convergence in $[\mathrm{BV}(\Omega)]^m$ because the dual space of BV is hard to characterize. However, it can be proved that BV is the dual of a separable space and, at least for sufficiently regular domains, the convergence stated in Equation (3.4) corresponds to weak-$*$ convergence in the usual sense.

Proposition 3.20. *Let $\{u_h\}_{h\in\mathbb{N}}\subset[\mathrm{BV}(\Omega)]^m$. Then $\{u_h\}_{h\in\mathbb{N}}$ weakly-$*$ converges to u in $[\mathrm{BV}(\Omega)]^m$ if and only if $\{u_h\}_{h\in\mathbb{N}}$ is bounded in $[\mathrm{BV}(\Omega)]^m$ and converges to u in $[\mathrm{L}^1(\Omega)]^m$.*

Definition 3.21 (Strict convergence). Let u, $u_h\in[\mathrm{BV}(\Omega)]^m$. We say that $\{u_h\}_{h\in\mathbb{N}}$ strictly converges in $[\mathrm{BV}(\Omega)]^m$ to u if $\{u_h\}_{h\in\mathbb{N}}$ converges to u in $[\mathrm{L}^1(\Omega)]^m$ and the variations $|Du_h|(\Omega)$ converges to $|Du|(\Omega)$ as $h\to\infty$.

Remark 3.22. The following formula is a distance in $[\mathrm{BV}(\Omega)]^n$, inducing strict convergence (and therefore weak-$*$ convergence, the opposite is not true):

$$d(u,v)=\int_\Omega |u-v|\,\mathrm{d}x + \big||Du|(\Omega)-|Dv|(\Omega)\big|.$$

Example 3.23. The function

$$u_h=\frac{\sin(hx)}{h}$$

weakly-$*$ converge to 0 in $\mathrm{BV}(0,2\pi)$, but the convergence is not strict because $|Du_h|(0,2\pi)=4$ for any $h\geq 1$.

Proposition 3.24. *If $\{u_h\}_{h\in\mathbb{N}}\subset[\mathrm{BV}(\Omega)]^m$ strictly converges to u, and $f:\mathbb{R}^{mn}\to\mathbb{R}$ is a continuous and positively 1-homogeneous function, we have*

$$\lim_{h\to\infty}\int_\Omega \varphi f\left(\frac{Du_h}{|Du_h|}\right)\mathrm{d}|Du_h| = \int_\Omega \varphi f\left(\frac{Du}{|Du|}\right)\mathrm{d}|Du|,$$

for any bounded continuous function $\varphi:\Omega\to\mathbb{R}$. As a consequence, the measures

$$f\left(\frac{Du_h}{|Du_h|}\right)|Du_h|\overset{*}{\rightharpoonup} f\left(\frac{Du}{|Du|}\right)|Du|, \quad \text{in } \Omega;$$

in particular,

$$|Du_h|\overset{*}{\rightharpoonup}|Du| \quad \text{and} \quad |Du_h^\alpha|\overset{*}{\rightharpoonup}|Du^\alpha|,\ \forall\alpha\in\{1,\dots,m\}.$$

3.3 Approximate continuity and differentiability

Since a BV function u of $n > 1$ variables doesn't have *good representatives* as in dimension 1, we need the notions of weak continuity and differiability.

Definition 3.25 (Approximate limit)**.** Let $u \in [\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$; we say that u has an *approximate limit* at $x \in \Omega$ if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \to 0} \fint_{B(x,\rho)} |u(y) - z| \, \mathrm{d}y = 0. \tag{3.5}$$

The set S_u of points where this property does not hold is called the *approximate discontinuity set*. For any $x \in \Omega \setminus S_u$ the vector z, uniquely determined by Equation (3.5), is called *approximate limit* of u at x and is denoted by $\widetilde{u}(x)$.

Remark 3.26. In the following we say that u is *approximately continuous* at x of $x \notin S_u$ and $\widetilde{u} = u(x)$, i.e. x is a Lebesgue point of u.

Aiming to recover the approximate discontinuity points, i.e. those corresponding to an approximate jump discontinuity between two values a and b along a direction ν, we introduce the following notation:

$$\begin{cases} B^+_\rho(x,\nu) = \{y \in B(X,\rho) : \langle y - x, \nu \rangle > 0\} \\ B^+_\rho(x,\nu) = \{y \in B(X,\rho) : \langle y - x, \nu \rangle < 0\} \end{cases}$$

and

$$u_{a,b,\nu}(y) = \begin{cases} a & \text{if } \langle y, \nu \rangle > 0 \\ b & \text{if } \langle y, \nu \rangle < 0 \end{cases}$$

for the two half balls contained in $B(x,\rho)$ determined by ν and for the function jumping between a and b along the hyperplane orthogonal to ν.

Definition 3.27 (Approximate jump points)**.** Let $u \in [\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ and $x \in \Omega$. We say that x is an *approximate jump point of u* if there exist a, $b \in \mathbb{R}^m$ and $\nu \in \mathbf{S}^{n-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0} \fint_{B^+_\rho(x,\nu)} |u(y) - a| \, \mathrm{d}y = 0, \quad \lim_{\rho \to 0} \fint_{B^-_\rho(x,\nu)} |u(y) - b| \, \mathrm{d}y = 0. \tag{3.6}$$

The triplet (a, b, ν), uniquely determined by Equation (3.6) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$. The set of approximate jump points is denoted by J_u.

Definition 3.28 (Approximate differentiability)**.** Let $u \in [\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ and let $x \in \Omega \setminus S_u$; we say that u is approximately differentiable at x if there exists a $m \times n$ matrix L such that

$$\lim_{\rho \to 0} \fint_{B(x,\rho)} \frac{|u(y) - \widetilde{u}(x) - L(y - x)|}{\rho} \, \mathrm{d}y = 0. \tag{3.7}$$

If u is approximately differentiable at x the matrix L, uniquely determined by Equation (3.7) is called the *approximate differential* of u at x and denoted by $\nabla u(x)$. The set of approximate differentiability points is denoted by $\mathscr{D}_u$.

Theorem 3.29 (Coarea formula in BV). *For any open set $\Omega \subset \mathbb{R}^n$ and $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ one has*

$$V(u,\Omega) = \int_{-\infty}^{\infty} P(\{x \in \Omega : u(x) > t\}; \Omega)\,\mathrm{d}t.$$

In particular, if $u \in \mathrm{BV}(\Omega)$, the set $\{u > t\}$ has finite perimeter in Ω for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ and

$$|Du|(B) = \int_{-\infty}^{\infty} |D\chi_{\{u>t\}}|(B)\,\mathrm{d}t, \quad Du(B) = \int_{-\infty}^{\infty} D\chi_{\{u>t\}}(B)\,\mathrm{d}t$$

for any Borel set $B \subset \Omega$.

Theorem 3.30 (Calderón-Zygmund). *Any function $u \in [\mathrm{BV}(\Omega)]^m$ is approximately differentiable at $\mathscr{L}^n$-almost every point of Ω. Moreover, the approximate differential ∇u is the density of the absolutely continuous part of Du, written as $D^a u$, with respect to $\mathscr{L}^n$.*

Definition 3.31 (Jump and Cantor parts). For any $u \in [\mathrm{BV}(\Omega)]^m$, the measures

$$D^j u = D^s u \llcorner J_u \quad \text{and} \quad D^c u = D^s u \llcorner (\Omega \setminus S_u)$$

are called respectively the *jump part* of the derivative and the *Cantor part* of the derivative.

Theorem 3.32 (Federer-Vol'pert). *Let $u \in [\mathrm{BV}(\Omega)]^m$: then the discontinuity set is $\mathscr{H}^{n-1}$-rectifiable and $\mathscr{H}^{n-1}(S_u \setminus J_u) = 0$. One has*

$$Du = \nabla u(x)\,\mathrm{d}x + (u_+(x) - u_-(x)) \otimes \nu_u(x)\,\mathrm{d}\mathscr{H}^{n-1} \llcorner J_u + D^c u$$

where $D^c u$, the Cantor part of Du with respect the Lebesgue measure, vanishes on any set A with $\mathscr{H}^{n-1}(A) < \infty$. In other words, for any $\varphi \in C^1_c(\Omega)$.

3.4 BV embedding

Theorem 3.33 (Convergence to the precise representative). *Let u be a function in $[\mathrm{BV}(\Omega)]^m$ and define the precise representative $u^* : \Omega \setminus (S_u \setminus J_u) \to \mathbb{R}^m$ of u to be equal to $\tilde{u}$ on $\Omega \setminus S_u$ and equal to $[u^+ + u^-]/2$ on J_u. Then the mollified functions $u * \rho_\varepsilon$ pointwise converge to u^* in its domain. In dimension 1, the precise representative is the* good representative.

Theorem 3.34. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected extension domain. Then*

$$\int_\Omega |u - u_\Omega|\,\mathrm{d}x \le C|Du|(\Omega), \quad \forall u \in \mathrm{BV}(\Omega),$$

for some real constant C depending only on Ω and where u_Ω of $u \in \mathrm{L}^1(\Omega)$ is the mean value defined as

$$u_\Omega = \fint_\Omega u(x)\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x.$$

The isoperimetric inequality in Equation (2.45) in conjuction with the coarea formula are used to prove the embedding Theorem of $\mathrm{BV}(\mathbb{R}^n)$ in $\mathrm{L}^{n/(n-1)}(\mathbb{R}^n)$.

Definition 3.35. Given $p \in [1,n]$ we define

$$p^* = \begin{cases} \dfrac{np}{n-1} & \text{if } p < N \\ \infty & \text{if } p = N, \end{cases}$$

under which $1^* = \infty$ if $n = 1$ and $n/(n-1)$ otherwise.

Theorem 3.36 (Embedding theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded extension domain. Then, the embedding* $\mathrm{BV}(\Omega) \hookrightarrow \mathrm{L}^{1^*}(\Omega)$ *is continuous and the embeddings* $\mathrm{BV}(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ *are compact for* $1 \le p < 1^*$.

Remark 3.37 (Poincaré inequality)**.** If Ω is a bounded connected extension domain, the continuity of the embedding of $\mathrm{BV}(\Omega)$ into $\mathrm{L}^{1^*}(\Omega)$ and Theorem 3.34, imply

$$||u - u_\Omega||_{\mathrm{L}^p(\Omega)} \le C|Du|(\Omega), \quad \forall u \in \mathrm{BV}(\Omega),\ 1 \le p < 1^*,$$

for some constant C depending only on Ω.

3.5 Compactness

Definition 3.38 (Extension domains)**.** An open set $\Omega \subset \mathbb{R}^n$ is an *extension domain* of $\partial\Omega$ is bounded and for any open set $A \supset \overline{\Omega}$ and any $m \ge 1$, there exists a linear and continuous extension operator $T : [\mathrm{BV}(\Omega)]^m \to [\mathrm{BV}(\mathbb{R}^n)]^m$ satisfying

- $Tu = 0$ a.e. in $\mathbb{R}^n \setminus A$ for any $u \in [\mathrm{BV}(\Omega)]^m$;
- $|DTu|(\partial\Omega) = 0$ for any $u \in [\mathrm{BV}(\Omega)]^m$;
- for any $p \in [1,\infty]$ the restriction of T to $[W^{1,p}(\Omega)]^m$ induces a linear continuous map between this space and $[W^{1,p}(\mathbb{R}^n)]^m$.

Proposition 3.39. *Any open set Ω with compact Lipschitz boundary is an extension domain.*

Theorem 3.40 (Compactness)**.** *Every sequence $\{u_h\}_{h\in\mathbb{N}} \subset [\mathrm{BV}_{\mathrm{loc}}(\Omega)]^m$ satisfying*

$$\sup\left\{ \int_A |u_h|\,\mathrm{d}x + |Du_h|(A) : h \in \mathbb{N} \right\} < \infty, \quad \forall A \subset\subset \Omega \text{ open},$$

admits a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ converging in $[\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ to $u \in [\mathrm{BV}_{\mathrm{loc}}(\Omega)]^m$. If Ω is a bounded extension domain and the sequence is bounded in $[\mathrm{BV}(\Omega)]^m$ we can say that $u \in [\mathrm{BV}(\Omega)]^m$ and that the subsequence weakly-$*$ converges to u.

Remark 3.41. Theorem 3.40 is very useful in connextion with variational problems with linear growth in the gradient. Since the Sobolev space $W^{1,1}$ has no similar compactness property, this provides also a justification for the introduction of BV spaces in calculus of variation.

Theorem 3.42 (Rellich's compactness theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, and let $\{u_h\}_{h\geq 1}$ be a sequence of functions in* $\mathrm{BV}(\Omega)$ *such that* $\sup_h \|u_h\|_{\mathrm{BV}(\Omega)} < \infty$*. Then there exists $u \in \mathrm{BV}(\Omega)$ and a subsequence $\{u_{h_k}\}_{k\geq 1}$ such that $u_{h_k} \to u$ (strongly) in $\mathrm{L}^1(\Omega)$ as $k \to \infty$.*

3.6 Function of Bounded Variation in one dimension

Definition 3.43 (Pointwise variation)**.** Let $a, b \in \overline{\mathbb{R}}$ with $a < b$ and $I = (a, b)$. For any function $u : I \to \mathbb{R}^m$ the *pointwise variation $pV(u,I)$* of u in I is defined by

$$\mathrm{pV}(u,I) = \sup\left\{\sum_{i=1}^{n-1} |u(t_{i+1}) - u(t_i)| : n \geq 2,\ a < t_1 < \cdots < t_n < b\right\}. \tag{3.8}$$

If $\Omega \subset \mathbb{R}$ is open, the pointwise variation $\mathrm{pV}(u,\Omega)$ is defined by $\sum_I \mathrm{pV}(u,I)$, where the sum runs along all the connected components of Ω.

Remark 3.44. The mapping $u \longmapsto \mathrm{pV}(u,\Omega)$ is lower semicontinuous with respect to pointwise convergence in I, being a supremum of continuous functionals. By additivity the same is true for $u \longmapsto \mathrm{pV}(u,\Omega)$, for any open set $\Omega \subset \mathbb{R}$.

Remark 3.45. Any function u with finite pointwise variation in an interval $I \subset \mathbb{R}$ is bounded, because its oscillation can be estimated with $\mathrm{pV}(u,I)$.

Clearly, $\mathrm{pV}(u,\Omega)$ is very sensitive to modifications of the values of u even at a single point. This suggest the following definition.

Definition 3.46 (Essential variation)**.** We define the *essential variation* $\mathrm{eV}(u,\Omega)$ of u in Ω as

$$\mathrm{eV}(u,\Omega) = \inf\left\{\mathrm{pV}(v,\Omega) :\ v = u\ \mathscr{L}^1\text{-a.e. in } \Omega\right\}. \tag{3.9}$$

Theorem 3.47. *For any $u \in [\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^m$ the infimum in Equation* (3.9) *is achieved and the variation $V(u,\Omega)$ coincides with the essential variation* $\mathrm{eV}(u,\Omega)$*.*

Remark 3.48. If $u \in [\mathrm{BV}(\Omega)]^m$, then, by Proposition 3.12, $V(u,\Omega) = |Du|(\Omega) < \infty$; since $V(u,\Omega) = \mathrm{eV}(u,\Omega)$, there exists $\overline{u}$, called *good representative*, in the equivalence class of u such that

$$\mathrm{pV}(\overline{u},\Omega) = \mathrm{eV}(u,\Omega) = V(u,\Omega).$$

Theorem 3.49 (Good representatives)**.** *Let $I = (a, b) \subset \mathbb{R}$ be an interval and $u \in [\mathrm{BV(I)}]^m$. Let A be the set of atoms of Du, i.e. $t \in A$ if and only if $Du(\{t\}) \neq 0$. Then, the following statements hold:*

- *there exists an unique* $c \in \mathbb{R}^m$ *such that*

$$u^l(t) = c + Du((a,t)), \quad u^r = c + Du((a,t]), \quad t \in I$$

are good representatives of u*, the left continuous one and the right continuous one. Any other function* $\overline{u} : I \to \mathbb{R}$ *is a good representative of* u *if and only if*

$$\overline{u}(t) \in \Big\{\theta u^l(t) + (1-\theta)u^r(t) : \theta \in [0,1]\Big\}, \quad \forall t \in I;$$

- *any good representative* $\overline{u}$ *is continuous in* $I \setminus A$ *and has a jump discontinuity at any point of* A*:*

$$\overline{u}(t_-) = u^l(t) = u^r(t_-) \quad \textit{and} \quad \overline{u}(t_+) = u^l(t_+) = u^r(t), \quad \forall t \in A.$$

- *any good representative* $\overline{u}$ *is differentiable at* $\mathscr{L}^1$*-a.e. point of* I*. The derivative* $\overline{u}'$ *is the density of* Du *with respect to* $\mathscr{L}^1$*.*

Corollary 3.50 (Monotone functions)**.** *Let* $u : (a,b) \to \mathbb{R}$ *be a monotone function. Then* u *is differentiable at* $\mathscr{L}^1$*-a.e.* $t \in (a,b)$ *and*

$$|u(b_-) - u(a_+)| \geq \int_a^b |u'(t)|\,\mathrm{d}t + \sum_{t \in \Gamma_u} |u(t_+) - u(t_-)|,$$

where Γ_u *is the discontinuity set of* u*.*

Theorem 3.51. *Let* $(a,b) \subset \mathbb{R}$ *be a bounded interval. Then the linear map*

$$(c,\mu) \longmapsto u(t) = c + \mu((a,t))$$

established an isomorphism between the Banach spaces $\mathbb{R}^m \times [\mathscr{M}(a,b)]^m$ *and* $[\mathrm{BV(a,b)}]^m$*.*

Remark 3.52. The equations defining u^l and u^r can be rephrased without mention of c:

$$u^l(s) - u^l(t) = Du([t,s)), \quad u^r(s) - u^r(t) = Du((t,s]), \quad a < t < s < b$$

and could be considered as the fundamental Theorem of Calculus in BV.

Definition 3.53 (Absolutely continuous functions)**.** Let $\Omega \subset \mathbb{R}$ be open and let $u \in [\mathrm{L}^1(\Omega)]^m$. We say that u is *absolutely continuous* in Ω if $u \in [\mathrm{BV}(\Omega)]^m$ and Du is absolutely continuous with respect to $\mathscr{L}^1$. The vector space of absolutely continuous functions in Ω coincides with the Sobolev space $[W^{1,1}(\Omega)]^m$.

Remark 3.54. In general, any measure μ on an open set $\Omega \subset \mathbb{R}$ can be split into three parts, an absolutely continuous one μ^a (with respect to $\mathscr{L}^1$), a purely atomic one μ^j, and a diffuse (i.e. without atoms) singular one μ^c. To obtain this decomposition we first denote by $A = \{t \in \Omega : \mu(\{t\}) \neq 0\}$ the set of atoms of μ (recall that A is at most countable), then we split μ into the absolutely continuous part μ^a and the singular part μ^s, given by the Radon-Nikodým Theorem, and we define $\mu^j = \mu^s \llcorner A$ and $\mu^c = \mu^s \llcorner (\Omega \setminus A)$. In this way we obtain

$$\mu = \mu^a + \mu^s = \mu^a + \mu^j + \mu^c.$$

This decomposition of μ is unique and since the measures μ^a, μ^j, μ^c are mutually singular we have also $|\mu| = |\mu^a| + |\mu^j| + |\mu^c|$. From the Theorem 1.55 we know that μ^s can also be represented by the restriction of μ to the $\mathscr{L}^1$-negligible set

$$S = \left\{ t \in \Omega : \lim_{\rho \to 0} \frac{|\mu|(B(t,\rho))}{\rho} = \infty \right\}$$

containing A, hence we can describe these three measures in a more constructive way:

$$\mu^a = \mu \llcorner (\Omega \setminus S), \quad \mu^j = \mu \llcorner A, \quad \mu^c = \mu \llcorner (S \setminus A).$$

According to this decomposition, we will say that $u \in \mathrm{BV}(\Omega)$ is a *jump* function if $Du = D^j u$, i.e. is a purely atomic measure, and we will say that u is a *Cantor* function if $Du = D^c u$, i.e. if Du is a singular measure without atoms.

Theorem 3.55 (Decomposition of BV(Ω) functions)**.** *Let $\Omega = (a,b) \subset \mathbb{R}$ be a bounded interval. Then, any $u \in [\mathrm{BV}(\Omega)]^m$ can be represented by $u^a + u^j + u^c$, where $u^a \in [W^{1,1}(\Omega)]^m$, u^j is a jump function and u^c is a Cantor function. The three functions are uniquely determined up to additive constants and*

$$\begin{aligned} |Du|(\Omega) &= |Du^a|(\Omega) + |Du^j|(\Omega) + |Du^c|(\Omega) \\ &= \int_a^b |\overline{u}'| \, \mathrm{d}t + \sum_{t \in A} |\overline{u}(t_+) - \overline{u}(t_-)| + |Du^c|(\Omega), \end{aligned}$$

where $\overline{u}$ is a good representative of u.

Remark 3.56. This decomposition of BV(Ω) functions is typical of the dimension one. No similar result is true for BV(Ω) functions of two or more variables. See Example 3.63 for more details.

Example 3.57. An example of a function with absolutely continuous derivative is given by any function $u \in W^{1,1}(\Omega)$ or, more obviously, $u \in C^1(\overline{\Omega})$. An example of a function with derivative a pure jump is given by χ_A, A a Caccioppoli set.

Example 3.58 (Jump function)**.** Given any sequence $\{d_h\}_{h \in \mathbb{N}} \subset (0,1)$, one can define

$$u(t) = \sum_{\{h : d_h < t\}} 2^{-h}.$$

The distributional derivative of u is the positive finite measure $\mu = \sum_h 2^{-h} \delta_{d_h}$, because $u(t) = \mu((0,t))$ for any $t \in (0,1)$. In general, the distributional derivative of a jump function can be reconstructed by left and right limits of a good representative, as Theorem 3.49 shows.

Example 3.59 (Cantor functions)**.** These functions (we are talking of good representatives) are continuous in their domain and differentiable, with 0 derivative, almost everywhere. This shows that, unlike absolutely continuous functions and jump functions, the derivative of Cantor functions can be seen only as measure, in the distributional sense, and cannot be recovered from the classical analysis of the pointwise behaviour of the function, based on concepts like left limit, right limit and derivative. The famous Cantor-Vitali function is an example of a function with derivative purely Cantorian. The function is obtained as follows: $\Omega = (0,1)$ and we let $u_0(t) = t$, and for any $n \geq 0$,

$$u_{h+1}(t) = \begin{cases} \frac{1}{2}u_h(3t) & 0 \leq t \leq \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} \leq t \leq \frac{2}{3} \\ \frac{1}{2}(u_h(3t-2)+1) & \frac{2}{3} \leq t \leq 1. \end{cases} \tag{3.10}$$

Then, one checks that

$$\sup_{(0,1)} |u_{h+1} - u_h| = \frac{1}{2} \sup_{(0,1)} |u_h - u_{h-1}| = \frac{1}{2^h} \times \frac{1}{6}$$

so that $\{u_h\}_{h\geq 1}$ is a Cauchy sequence and converges uniformly to some function u. This function is constant on each interval the complement of the triadic Cantor set, which has zero measure in $(0,1)$. Hence, almost everywhere, its classical derivative exists and is zero. One can deduce that the derivative Du is singular with respect to Lebesgue's measure. On the other hand, it is continuous as a uniform limit of continuous functions, hence Du has no jump part. In fact, $Du = D^c u$, which in this case, is the measure $\mathscr{H}^{\ln 2/\ln 3} \llcorner D^c / \mathscr{H}^{\ln 2/\ln 3}(D^c)$.

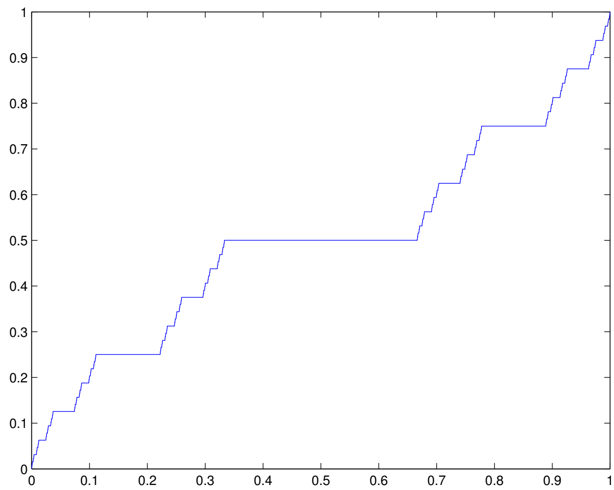


Figure 7: Cantor-Vitali function, also known as *devil's staircase*.

The terminology of the decomposition is justified by the Cantor-Vitali function (see Example 3.59), whose distributional derivative has no jump part and no absolutely continuous part. We also call

$D^a u + D^c u$ the *diffusive* part of the derivative and denote it by $\widetilde{D}u$. This decomposition of Du has several motivations:

- theoretical: the different behaviour of the diffusive part and the jump part under left composition with Lipschitz mapping;
- practical: we are interested in the integral functionals on BV and we want to identify the absolutely continuous and the singular part of the energy (in some case a Cantor part).

3.7 Special functions of Bounded Variation

The Special functions of Bounded Variation have been singled out by E. De Giorgi and L. Ambrosio in [De Giorgi and Ambrosio(1988)] as good candidates for setting a wide class of variational problems where both volume and surface energies are involved.

Definition 3.60 (SBV space)**.** We say that $u \in \mathrm{BV}(\Omega)$ is a special function with bounded variation, and we write $u \in \mathrm{SBV}(\Omega)$, if the Cantor part of its derivative $D^c u$ is zero. So we obtain

$$Du = D^a u + D^j u = \nabla u \mathscr{L}^n + (u^+ - u^-)\nu_u \mathscr{H}^{n-1} \llcorner J_u, \quad \forall u \in \mathrm{SBV}(\Omega). \tag{3.11}$$

Remark 3.61. $\mathrm{SBV}(\Omega)$ is a *proper* subspace of $\mathrm{BV}(\Omega)$ if $\Omega \subset \mathbb{R}$: in fact, the Cantor-Vitali function (3.10) belongs to $\mathrm{BV}((0,1)) \setminus \mathrm{SBV}((0,1))$. Considering Cantor-Vitali like functions which depend only on one variable it us easy to realize that $\mathrm{SBV}(\Omega)$ is a proper subspace of $\mathrm{BV}(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$. Unfortunately ∇u and (u^+, u^-, ν_u) are not sufficient to build the distributional derivative of a general $\mathrm{BV}(\Omega)$ function.

Remark 3.62. $\mathrm{W}^{1,1}(\Omega) \subset \mathrm{SBV}(\Omega) \subset \mathrm{BV}(\Omega)$ and both inclusions are strict. In fact, $\mathrm{SBV}(\Omega)$ contains bounded *piecewise Sobolev* functions in a very weak sense. For example, if $u = \chi_A$ and $|A| < \infty$, $0 < P(A;\Omega) < \infty$ then $u \in \mathrm{SBV}(\Omega)$ but u is not a Sobolev function because $Du = \nu_A \mathscr{H}^{n-1} \llcorner \partial^* A$. So,

$$u \in \mathrm{W}^{1,1}(\Omega) \iff \mathscr{H}^{n-1}(S_u) = 0, \quad \forall u \in \mathrm{SBV}(\Omega).$$

Example 3.63 (Square root example)**.** We show that the decomposition (3.11) doesn't hold for dimensions higher that 1. Let $S = (-\infty, 0) \times \{0\} \subset \mathbb{R}^2$ be the left x axis and let $u : \mathbb{R}^2 \setminus S \to \mathbb{R}$ be defined, in polar coordinates $(\rho, \theta) \in (0,\infty) \times (-\pi, \pi)$, by $\sqrt{\rho}\sin(\theta/2)$. Then, it is easy to check that $u \in \mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^2)$, with $S_u = J_u = S \setminus \{0\}$. If it were possible to decompose u in the form $u = u^a + u^j$ with $u^a \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ and $D^a u^j = 0$, we would obtain that $\nabla(u - u^a) = \nabla u^j = 0$. Since $(u - u^a) \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^2 \setminus S)$ and $\mathbb{R}^2 \setminus S$ is connected, we conclude that $u - u^a$ is (equivalent to a) constant. Hence, $u = u^a + c \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, a contradiction.

Since, by definition, $D^c u = D^s u \llcorner (\Omega \setminus S_u)$, we can say that u belongs to $\mathrm{SBV}(\Omega)$ if and only if $D^s u$ is concentrated on S_u. More generally:

Proposition 3.64. *Any $u \in \mathrm{BV}(\Omega)$ belongs to $\mathrm{SBV}(\Omega)$ if and only if $D^s u$ is concentrated on a Borel set σ-finite with respect to $\mathscr{H}^{n-1}$.*

Theorem 3.65. *$\mathrm{SBV}(\Omega)$ is a closed subspace of $\mathrm{BV}(\Omega)$.*

Proof. If I is finite or countable, $u_i \in \mathrm{SBV}(\Omega)$ for any $i \in I$ and $\sum_{i\in I} u_i$ converges to $u \in \mathrm{BV}(\Omega)$ in the BV norm, then $Du = \sum_i Du_i$. Since $\sum_i D^a u_i$ is absolutely continuous with respect to $\mathscr{L}^n$ and $\sum_i D^s u_i$ is singular, we have

$$D^a u = \sum_{i\in I} D^a u_i, \quad D^s u \sum_{i\in I} D^s u_i,$$

and since $D^s u$ is concentrated on $\bigcup_i S_{u_i}$, we conclude that $u \in \mathrm{SBV}(\Omega)$. This proves that $\mathrm{SBV}(\Omega)$ is a closed subspace of $\mathrm{BV}(\Omega)$. □

Proposition 3.66. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $K \subset \mathbb{R}^n$ closed and assume that $\mathscr{H}^{n-1}(K \cap \Omega) < \infty$. Then, any function $u : \Omega \to \mathbb{R}$ that belongs to $\mathrm{L}^\infty(\Omega \setminus K) \cap W^{1,1}(\Omega \setminus K)$ belongs also to $\mathrm{SBV}(\Omega)$ and satisfies $\mathscr{H}^{n-1}(S_u \setminus K) = 0$.*

Theorem 3.67 (Closure of SBV). *Let $\varphi : [0,\infty) \to [0,\infty)$, $\theta : [0,\infty) \to [0,\infty)$ be lower semicontinuous increasing functions and assume that*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty, \quad \lim_{t\to 0} \frac{\theta(t)}{t} = \infty.$$

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\{u_h\}_{h\in\mathbb{N}} \subset \mathrm{SBV}(\Omega)$ such that

$$\sup_h \left\{ \int_\Omega \varphi(|\nabla u_h|)\,\mathrm{d}x + \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|)\,\mathrm{d}\mathscr{H}^{n-1} \right\} < \infty. \tag{3.12}$$

If $\{u_h\}_{h\in\mathbb{N}}$ weakly-$*$ converges in $\mathrm{BV}(\Omega)$ to u, then $u \in \mathrm{SBV}(\Omega)$, the approximate gradients ∇u_h weakly converge to ∇u in $[\mathrm{L}^1(\Omega)]^n$, $D^j u_h$ weakly-$*$ converge to $D^j u$ in Ω and

$$\int_\Omega \varphi(|\nabla u|)\,\mathrm{d}x \le \liminf_{h\to\infty} \int_\Omega \varphi(|\nabla u_h|)\,\mathrm{d}x, \qquad \textit{if } \varphi \textit{ is convex},$$

$$\int_{J_u} \theta(|u^+ - u^-|)\,\mathrm{d}\mathscr{H}^{n-1} \le \liminf_{h\to\infty} \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|)\,\mathrm{d}\mathscr{H}^{n-1}, \qquad \textit{if } \theta \textit{ is concave}.$$

Theorem 3.68 (Compactness of SBV). *Let φ, θ, Ω as in Theorem 3.67. Let $\{u_h\}_{h\in\mathbb{N}} \subset \mathrm{SBV}(\Omega)$ be satisfying Equation* (3.12) *and assume, in addition, that $||u_h||_\infty$ is uniformly bounded in h. Then, there exists a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ weakly-$*$ converging in $\mathrm{BV}(\Omega)$ to $u \in \mathrm{SBV}(\Omega)$.*

3.8 Boundary Trace

In many situations one needs to study functions arising form *cutting* and *pasting* two functions $u, v \in [\mathrm{BV}(\Omega)]^m$; denoting the cut region by A, one needs to know whether $w = u\chi_A + v\chi_{\Omega\setminus A}$ is still in $[\mathrm{BV}(\Omega)]^m$, and if this is the case a formula for Dw is needed. We prove that w is a BV function if A is a set of finite perimeter and the difference $u^+_{\partial^*A} - v^-_{\partial^*A}$ between the *interior trace* of u and the *exterior trace* of v is summable on ∂^*A. This summability condition is of course satisfied if u and v are globally bounded and if A is sufficiently regular.

Theorem 3.69 (Traces on interior rectifiable sets). *Let u be a function in $[\mathrm{BV}(\Omega)]^m$ and let $\Gamma \subset \Omega$ be a countably $\mathscr{H}^{n-1}$-rectifiable set oriented by ν. Then, for $\mathscr{H}^{n-1}$-almost every $x \in \Gamma$ there exist $u^+_\Gamma(x)$, $u^-_\Gamma(x)$ in $\mathbb{R}^m$ such that*

$$\begin{cases} \displaystyle\lim_{\rho\to 0} ⨍_{B^+_\rho(x,\nu(x))} |u(y) - u^+_\Gamma(x)|\,\mathrm{d}y = 0 \\ \displaystyle\lim_{\rho\to 0} ⨍_{B^-_\rho(x,\nu(x))} |u(y) - u^-_\Gamma(x)|\,\mathrm{d}y = 0. \end{cases}$$

Moreover, $Du \llcorner \Gamma = \left(u^+_\Gamma - u^-_\Gamma\right) \otimes \nu \mathscr{H}^{n-1} \llcorner \Gamma$.

Theorem 3.70. *Let $u, v \in [\mathrm{BV}(\Omega)]^m$ and let A be a set of finite perimeter in Ω, with $\partial^*A \cap \Omega$ oriented by ν_A. Let $u^+_{\partial^*A}$, $v^-_{\partial^*A}$ be given for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^*A \cap \Omega$ by Theorem 3.69. Then*

$$w = u\chi_A + v\chi_{\Omega\setminus A} \in [\mathrm{BV}(\Omega)]^m \iff \int_{\partial^*A\cap\Omega} |u^+_{\partial^*A} - v^-_{\partial^*A}|\,\mathrm{d}\mathscr{H}^{n-1} < \infty.$$

If $w \in [\mathrm{BV}(\Omega)]^m$, the measure Dw is representable by

$$Dw = Du \llcorner A^1 + (u^+_{\partial^*A} - v^-_{\partial^*A}) \otimes \nu_A \mathscr{H}^{n-1} \llcorner (\partial^*A \cap \Omega) + Dv \llcorner A^0.$$

Example 3.71. Let $u_1, u_2 \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and let A be a set of finite perimeter in Ω. Then, Theorem 3.70 implies that the function $u = u_1\chi_A + u_2\chi_{\Omega\setminus A}$ belongs to $\mathrm{SBV}(\Omega)$ and satisfies

$$Du = \left[\nabla u_1\chi_A + \nabla u_2\chi_{\Omega\setminus A}\right]\mathscr{L}^n + (\widetilde{u}_1 - \widetilde{u}_2)\nu_A\mathscr{H}^{n-1} \llcorner (\Omega \cap \partial^*A).$$

In fact, since S_{u_1} and S_{u_2} are both $\mathscr{H}^{n-1}$-negligible, the approximate limits $\widetilde{u}_1$ and $\widetilde{u}_2$ respectively coincide $\mathscr{H}^{n-1}$-a.e. on ∂^*A with the interior trace $(u_1)^+_{\partial^*A}$ and the exterior trace $(u_2)^-_{\partial^*A}$ given by Theorem 3.69. Assuming that $\widetilde{u}_1 - \widetilde{u}_2$ is $\mathscr{H}^{n-1} \llcorner (\Omega \cap \partial^*A)$ summable, the L^∞ assumption on u_1, u_2 can be dropped.

3.9 Caccioppoli Partitions

Definition 3.72 (Caccioppoli partition). Let $\Omega \in \mathbb{R}^n$ be an open set and $I \subset \mathbb{N}$; we say that a partition $\{A_i\}_{i\in\mathbb{N}}$ of Ω is a *Caccioppoli partition* if

$$\sum_{i\in I} P(A_i;\Omega) < \infty.$$

We say that a Caccioppoli partition $\{A_i\}_{i\in I}$ is ordered if $|A_i| \geq |A_j|$ whenever $i \leq j$. Given a $\mathcal{H}^{n-1}$-rectifiable set $K \subset \Omega$, we say that a Caccioppoli partition $\{A_i\}_{i\in I}$ of A is subordinated to K if, $\forall i \in I$, $\partial^* A_i \subset K$ up to a $\mathcal{H}^{n-1}$-negligible set.

Theorem 3.73 (Compactness of Caccioppoli partition). *Let $\{A_{i,h}\}_{i\in I}$, with $h \in \mathbb{N}$, be Caccioppoli partitions of a bounded open set Ω with Lipschitz boundary such that*

$$\sup_{i\in I}\left\{P(A_{i,h};\Omega) : h \in \mathbb{N}\right\} < \infty.$$

Then, if either I is finite or the partitions are ordered, there exists a Caccioppoli partition $\{A_i\}_{i\in I}$ and a subsequence $\{h_k\}_{k\in I}$ such that $\{A_{i,h_k}\}_{i\in I}$ converges in measure to A_i for any $i \in I$.

Definition 3.74 (Piecewise constant functions). We say that $u : \Omega \to \mathbb{R}^m$ is piecewise constant in Ω if there exists a Caccioppoli partition $\{A_i\}_{i\in I}$ of Ω and a collection $\{t_i\}_{i\in I} \subset \mathbb{R}^m$ such that

$$u = \sum_{i\in I} t_i \chi_{A_i}.$$

Theorem 3.75 (Characterization of piecewise constant functions). *Let $u \in [\mathrm{L}^\infty(\Omega)]^m$. Then, u is (equivalent to) a piecewise constant function if and only if $u \in [\mathrm{SBV}(\Omega)]^m$, Du is concentrated on S_u and $\mathcal{H}^{n-1}(S_u) < \infty$. Moreover, denoting by $\{A_i\}_{i\in I}$ the level sets of u and t_i the associated values, i.e. $A_i = \{x \in \Omega : u(x) = t_i\}$, we have $u = \sum_i t_i \chi_{A_i}$, $\partial^* A_i \subset S_u$ for all i up to $\mathcal{H}^{n-1}$ negligible sets and $2\mathcal{H}^{n-1}(S_u) = \sum_{i\in I} P(A_i;\Omega)$.*

Theorem 3.76 (Compactness of piecewise constant functions). *Let Ω be a bounded open set with Lipschitz boundary and let $\{u_h\}_{h\in\mathbb{N}} \subset [\mathrm{SBV}(\Omega)]^m$ be a sequence of piecewise constant functions such that*

$$||u_h||_\infty + \mathcal{H}^{n-1}(S_{u_h}) \textit{ is uniformly bounded.}$$

Then, there exists a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ converging in L^1 to a piecewise constant function.

3.10 Piecewise roto-translations

Definition 3.77 (Piecewise roto-translations). We say that $u \in \mathrm{SBV}(\Omega)$ is a piecewise roto-translation in Ω if there exists a Caccioppoli partition $\{A_i\}_{i\in I}$ of Ω and $R_i \in \mathrm{SO(n)}$ (the special orthogonal group of all rotations in $\mathbb{R}^n$), $t_i \in \mathbb{R}^n$, $i \in I$, such that, for a.e. $x \in \Omega$,

$$u(x) = \sum_{i \in I} (R_i x + t_i)\chi_{A_i}(x).$$

Theorem 3.78 (Compactness of piecewise roto-translations). *Let* $\{u_h\}_{h\in\mathbb{N}} \subset \mathrm{SBV}(\Omega)$ *be a sequence of piecewise roto-translations associated with Caccioppoli partitions* $\{A_{i,h}\}_{i\in I}$ *such that*

$$||u_h||_\infty + \sum_{i \in I_h} P(A_{i,h};\Omega) \text{ is uniformly bounded.}$$

Then, there exists a subsequence $\{u_{h_k}\}_{k\in\mathbb{N}}$ *converging in* L^1 *to a piecewise roto-translation.*

Theorem 3.79 (Chambolle, Giacomini, Ponsiglione). *If* $u \in \mathrm{SBV}(\Omega)$ *is such that* $\mathcal{H}^{n-1}(S_u) < \infty$ *and* $\nabla u(x) \in \mathrm{SO(n)}$ *for* $\mathcal{L}^n$*-a.e.* $x \in \Omega$*, then there exists a Caccioppoli partition* $\{A_i\}_{i\in I}$ *subordinated to* S_u *such that*

$$u = \sum_{i \in I} (R_i x + t_i)\chi_{A_i}$$

where $R_i \in \mathrm{SO(n)}$ *and* $t_i \in \mathbb{R}^n$.

Remark 3.80. Theorem 3.79 provides an interesting characterization of piecewise roto-translations that is useful to study the links between rigidity and elastic energy in the context of fracture mechanics. This result motivates the use of SBV and the Caccioppoli partitions as the appropriate tools for representing and manipulating piecewise roto-translations.

CHAPTER 4

The Inpainting problem

This chapter aims to motivate the previous study of BV space function theory, showing that the functional and geometric properties of this space are useful to model real problems. These functions appear as L^1 limits of Sobolev (and even more regular) functions when dealing with integral functionals with a linear growth in the gradient. Meanwhile, the level sets of a BV function are all (generally) sets of finite perimeter. Moreover, the SBV space, a special subset of BV space function, appears as the natural setting where to study variational models with both volume and surface energy density to be taken into account. To deepen this topics we refer to [Ambrosio *et al.*(2000), Maggi(2012), Giusti(1977)].

Example 4.1 (Sets with prescribed curvature)**.** The simplest problem where volume and surface energies compete is probably the so called prescribed curvature problem:

$$\min_{A}\left\{\int_A g(x)\,\mathrm{d}x+\mathscr{H}^{n-1}(\partial A) : a\subset\mathbb{R}^n\right\},$$

where $g\in L^1(\mathbb{R}^n)$ is given. In this problem, if $g(x)<0$ in some region F, the two terms can have opposite sign, and, if F is not too irregular, it may be convenient to include F in A to decrease the value of the functional. The terminology for this problem can be explained through the first variation: if g is continuous at a regular point $x\in\partial A$, and A minimises the functional, then the equation

$$\mathbf{H}(x)=g(x)\nu_A(x)$$

holds, where $\mathbf{H}$ is the mean curvature vector of ∂A and ν_A is the outer normal to A. To solve this problem, the classical framework to be used comes from the sets of finite perimeter.

Example 4.2 (Optimal partitions)**.** A generalization of the prescribed mean curvature problem, is the optimal partition problem: given $\Omega\in\mathbb{R}^n$ and $g\in L^\infty(\Omega)$, one looks for the following:

$$\min_{K,u}\left\{\mathscr{H}^{n-1}(K\cap\Omega)+\alpha\int_{\Omega\setminus K}|u-g|^2\,\mathrm{d}x\right\}$$

among all the closed sets $K \subset \mathbb{R}^n$ and all the functions u that are constant in the connected components of $\Omega \backslash K$. This minimum problem corresponds to the best piecewise constant approximation of g, with control (whose strenght depends on α) on the total area of the discontnuity set K, and is also interesting in image segmentation. Notice that if K is given then obviously the value of u in each connected component is the mean value of g therein, and giving u in turn determines K as discontinuity set, so that the unknown variables K, u can reduce the only one, and the above problem can be easily rephrased in SBV(Ω) looking for the

$$\min_u \left\{ \mathscr{H}^{n-1}(S_u) + \alpha \int_\Omega |u-g|^2 \,\mathrm{d}x \right\}$$

among all the piecewise constant functions $u \in \mathrm{SBV}(\Omega)$.

Example 4.3 (Mumford-Shah image segmentation problem)**.** Assume that a bounded open set $\Omega \subset \mathbb{R}^n$, a function $g \in \mathrm{L}^\infty(\Omega)$ and a strictly positive parameters α, β are given; the Mumford-Shah functional is defined by

$$J(K,u) = \int_{\Omega\backslash K} \left[|\nabla u|^2 + \alpha(u-g)^2 \right] \mathrm{d}x + \beta \mathscr{H}^{n-1}(K \cap \Omega).$$

The problem is to minimize J in the set of admissible pairs

$$\mathscr{A} = \left\{ J(K,u) : K \subset \overline{\Omega} \text{ closed}, u \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus K) \right\}.$$

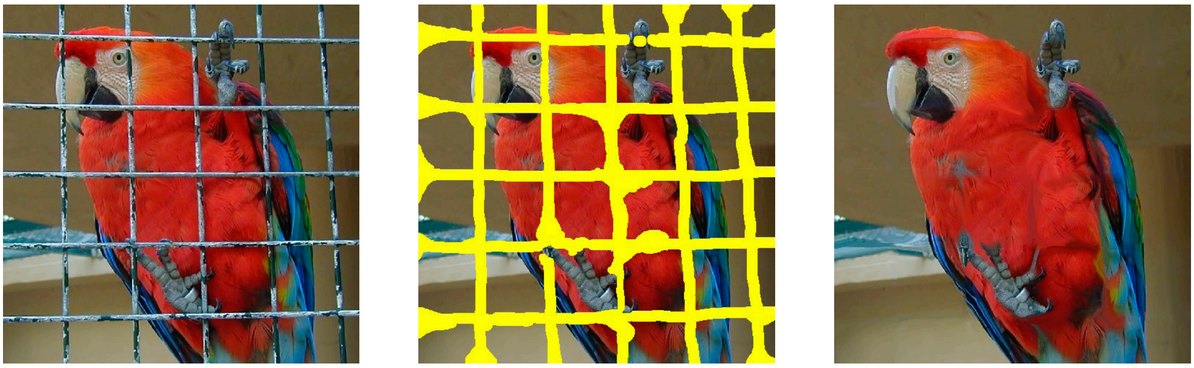
Example 4.4 (Rudin-Osher-Fatemi for image denoising and deblurring)**.**

$$\min_u \lambda \int_\Omega |Du| \,\mathrm{d}x + \frac{1}{2} \int_\Omega |u(x) - g(x)|^2 \,\mathrm{d}x.$$

Many other problems arising in real applications can be set in BV space.

4.1 State of the art

Among all possible applications of BV theory, one of the most recent and challenging problem is refereed to *inpainting* problems. This is the topic we choose to depeen. The *image inpainting* problem (or *image completion* or *disocclusion*) for image processing can be stated as the situation in which a corrupted image is given and we want to restore the data missing with the only available data captured outside the region to be *inpainted*. An example can be found in Figure 8. Today, *inpainting* is a very common problem in film restoration and image retouching, so many software in image editing have specific plug-in or controls to manage with this problem. The term *digital inpainting* was introduced into image processing by Bertalmio, Sapiro, Caselles and Ballester in [Ballester *et al.*(2001)] and it is a very famous word among museum conservators and restoration artists: this is refereed to the practice of retouching or recovering damaged ancient paintings. The goal is to remove the cracks or recover the missing patches in an undetectable manner. The authors were the first to apply the PDE method to inpainting, introducing an innovative third order PDE.



(a) Original. (b) Inpainting domain marked. (c) Result.

Figure 8: Example of inpainting problem from [Bornemann and März(2007)].

Remark 4.5. Inpainting problem is essentially an an *interpolation* problem where the degree of the function to be interpolated is unknown. What makes image interpolation highly non trivial is the complexity of image functions. For example, in 1D, we can know the values (or even the derivatives) of a function f at $a-h$ and $a+h$. If f is smooth, then as $h \to 0$, we can apply smooth interpolants such as Lagrange or Hermite's to infer the values of f on $(a-h, a+h)$ with certain guaranteed degree of precision. But for a BV function f, all such smooth interpolants fail to work properly no matter how small h is, since a *widhless* jump can always occur in $(a-h, a+h)$. This is speciality of BV functions: by the TV (total variation) Radon measure, a single point is allowed to have nonzero mass, which makes the corrisponding interpolation problem ill-posed in general.

Remark 4.6. The aim of inpainting is not to recover the true occluded background but to create a new one which looks natural to a human observer: that's way the inpainting problem is related to a psychology problem called *amodal completion*, or *illusory contours*. For example, in Kanizsa's Triangle of Figure 9, spatially separate fragments give the impression of illusory contours of a triangle.

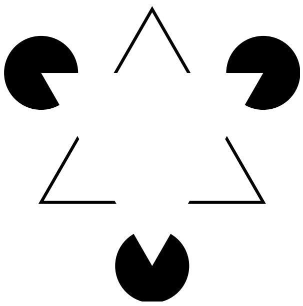


Figure 9: Kanizsa's Triangle.

Some attempts has been proposed to solve inpainting problem: for example a *geometric-oriented* approach was firstly proposed in [Masnou and Morel(1998)], from which Figure 10 is taken, while *texture-oriented* approach is the core of [Efros and Leung(1999)].

Geometry oriented. The main idea is to model images as functions with some degree of smoothness, expressed in terms of its total variation or the curvature of its level lines. Essentially, the boundary data of the inpainting domain is used to interpolate and predict the geometric structure of the image where unknown, providing a *local* methods, based on solving PDEs. These methods show good performance in propagating smooth level lines or gradients, but fail in the presence of texture.

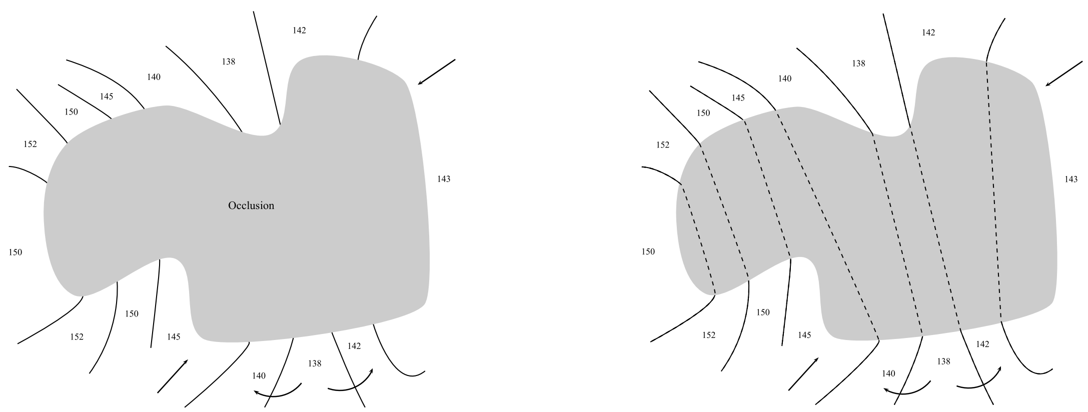


Figure 10: Occlusion with a possible connection of level lines by [Masnou and Morel(1998)].

Texture oriented. These methods model the texture as a two dimensional probabilistic *graphical model*, in which the value of each pixel is conditioned by its neighbourhood. The value of each target pixel x is copied from the center of a (square) patch in the sample image, chosen to match the available portion of the patch centred at x. These methods are *non-local* and are called *exemplar-based*: to determine the value at x, the whole image is scanned searching for a matching patch. The exemplar-based approach can be stated as finding a *correspondence map*

$$\varphi : O \to O^c$$

which assigns to each location x in the inpainting domain O (a subset of the image domain Ω, usually a rectangle in $\mathbb{R}^2$) a corresponding location $\varphi(x) \in O^c = \Omega \setminus O$, where the image is known [Demanet *et al.*(2003)]. The filling-in strategy can be regarded as a *greedy* procedure (each hole pixel is visited only once) for computing a *correspondence map*, but this is very sensitive to the order in which the pixels are processed. This motivates the introduction of a variational formulation for inpainting problem as the minimization of an energy functional in which the unknown variable is the correspondence map itself,

$$E(\varphi)=\int_{O}\int_{\Omega_p}|\widehat{u}(\varphi(x+h))-\widehat{u}(\varphi(x)+h)|^2\,\mathrm{d}h\,\mathrm{d}x, \tag{4.1}$$

where Ω_p is the patch domain, centred at $(0,0)$, and $\widehat{u}$ is the known image defined in O^c. The unknown image is computed as $u(x)=\widehat{u}(\varphi(x))$, for $x\in O$. Thus, φ should map a pixel x and its neighbours in such a way that the resulting patch is close to the one centred at $\varphi(x)$. Unfortunately, the energy (4.1) is highly non-convex and no effective way to minimize it is known [Aujol *et al.*(2010)].

It is possible to introduce a relaxation of energy (4.1) considering the correspondence map as auxiliary variable and the unknown equation is now determined as part of optimization process with an alternating minimization scheme: this removes the constraint $u(x)=\widehat{u}(\varphi(x))$:

$$E(u,\varphi)=\int_{\widetilde{O}}\int_{\widetilde{\Omega}_p}|u(x+h)-\widehat{u}(\varphi(x)+h)|^2\,\mathrm{d}h\,\mathrm{d}x,$$

where $\widetilde{O}=O+\Omega_p$ refers to the set of centres of patches that intersect the inpainting domain O. We will see later that this non-convex energy converges to a critical point.

In denoising and superresolution problems, the pixel values are estimated from many image locations and this is equivalent to replace the correspondence map with a weight function

$$w:\Omega\times\Omega\to\mathbb{R},$$

with Ω being the image domain (usually a rectangle in $\mathbb{R}^2$). For each x, $w(x,\cdot)$ weights the contribution of each image location to the estimation of x. In this context, [Gilboa and Osher(2007)] proposed the following functional for non-local means denoising method:

$$E_w(u)=\int_{\Omega}\int_{\Omega}w(x,y)(u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x,$$

where the weights w are considered as known and fixed through all iterations: this is not the case of inpainting problem because weights are not available or directly linked with the image data known. In this sense, the weights w are now considered as a variable of the problem and updated adaptively at each iteration. Heuristically, one can say that the optimal similarity weights w converge to $\delta(y-\varphi(x))$ where $\varphi:\widetilde{O}\to\widetilde{O}^c$ is the optimal correspondence map, with $y\in\widetilde{O}^c$ and $x\in\widetilde{O}$.

Other approaches. Other techniques have been proposed to solve the inpainting problem. In [Starck *et al.*(2005)], two appropriate dictionaries are used: one for the representation of textures, and the one for the natural scene parts assumed to be piecewise-smooth which produce very good results for *sparse* inpainting domains. In [Hays and Efros(2007)], the image completion is performed with an huge database of photographs gathered from the Web: the algorithm patches up holes in images by finding similar regions in the database that are not only seamless but also semantically valid.

We are now introducing some details about geometric and exemplar-based approach. Firstly, we present the inpainting problem modelled with curvature and level lines of surrounding data by [Chan *et al.*(2002)], then we present the most recent works [Arias *et al.*(2011), Arias *et al.*(2012)] and the related results in order to get the state of the art in this important research field.

4.2 A geometry oriented approach: the Euler's elastica

In 1744, Euler obtained the energy (4.2) studying the steady shape of a thin and torsion-free rod under external forces: that's how *Euler's elastica* born.

Definition 4.7. *Euler's elastica* is the equilibrium curve of the elasticity energy:

$$E_2[\gamma] = \int_\gamma (a + b\kappa^2)\,\mathrm{d}s, \tag{4.2}$$

where $\mathrm{d}s$ denotes the arc length element, $\kappa(s)$ the scalar curvature and a and b two positive constant weights.

Remark 4.8. Since both arc length and curvature are intrinsic geometric features of a curve, the elastica energy naturally extends to the curves living on a general Riemannian manifold M. If M is embedded in a Euclidean space $\mathbb{R}^n$, then a curve γ on M can be expressed by the embedded coordinates

$$s \rightarrow \vec{x}(s) = (x_1(s), \ldots, x_n(s)).$$

Then, $\vec{t} = \partial\vec{x}/\partial s$ is the tangent and $\prod_{\vec{x}} \partial\vec{t}/\partial s = \kappa\vec{n}$ defines the curvature and $\prod_{\vec{x}}$ is the orthogonal projection from $T_{\vec{x}}\mathbb{R}^n$ to $T_{\vec{x}}M$. For a general Riemannian manifold M, the intrinsic derivative $\partial\vec{t}/\partial s$ is defined by the Levi-Civita connection or covariant derivative.

By Calculus of Variation, from the energy formula (4.2), an elastica must satisfy the forth-order equation:

$$2\kappa''(s) + \kappa^3(s) = \frac{a}{b}\kappa(s),$$

or, more generally if the elastica lives on a Riemannian surface, there will be an extra term due to the curving of the surface (the Gaussian curvature of the surface G):

$$2\kappa''(s) + \kappa^3(s) + 2G(s)\kappa(s) = \frac{a}{b}\kappa(s),$$

Remark 4.9. Birkhoff and De Boor shown in [Birkhoff and de Boor(1965)] the link between elastica and computer vision as the interpolation capability of elastica: such *nonlinear* splines, like classical polynomial splines, are useful tools to complete the broken or occluded edges of objects in the 2D projection of a 3D scene.

Since digital inpainting is a sort of interpolation in 2D domain, we can start looking what an image u_0 looks like locally with a simple local patch D missing: we try to inpaint $u_0|_D$ based on the available information surrounding D. By checking the boundary data along D we can determine which *class* $u_0|_D$ belongs to (a *class* is a type of patch based on the complexity of expected local edge, see Figure 11), based on interpolation of boundary end points by elastica, instead of straight line segments.

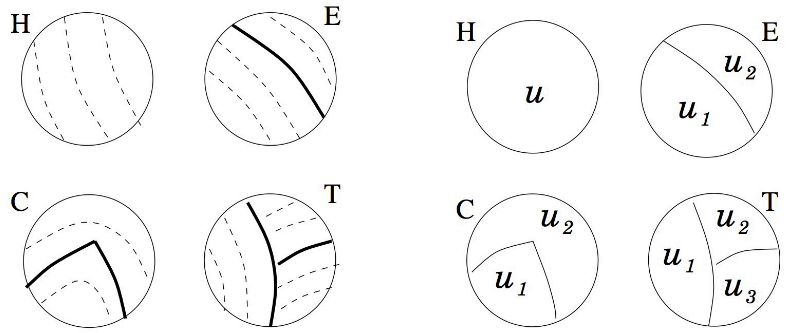


Figure 11: Four different classes of situations from [Chan *et al.*(2002)].

The first step is to inpaint the missing edges (smooth, corner or T-junction) to reduce class E, C, T to class H (more details in [Chan *et al.*(2002)]). Along the boundary, each end point can be represented by $(p, \vec{n})$, with p denoting its position and $\vec{n}$ the normal to the edge, which can be computed from the available image outside the inpainting domain D.

After the feature edges have all been interpolated, all four classes of local image inpaintings are essentially reduced to the inpainting of class H, the homogeneous patches. Such patches can also be inpainted by having the broken isophotes interpolated by elasticas *one by one* from the boundary information (like the algorithm proposed in [Masnou and Morel(1998)]).

Generically, one can assume that the missing smooth patch $u_0|_D$ is *regular* in the sense that it lies close to a regular point where ∇u_0 is non-zero (or by first applying a small step of Gaussian diffusion). Thus the isophotes of u_0 on D are well defined and distinguishable, and each Γ_λ is uniquely labelled by its gray level $u_0 \equiv \lambda$.

The trace of each Γ_λ on the boundary tells the coupling rule of boundary pixels. Suppose $p_1, p_2 \in \partial\Omega$ share the same gray level λ, and the normals computed from the available image data outside D are $\vec{n}_1, \vec{n}_2$. Then we inpaint the λ-isophote Γ_λ by elastica Γ'_λ:

$$\Gamma'_\lambda = \underset{\gamma_\lambda \vdash ((p_1,\vec{n}_1),(p_2,\vec{n}_2))}{\arg\min} \int_{\gamma_\lambda} (a + b\kappa^2)\,\mathrm{d}s = \underset{\gamma_\lambda \vdash ((p_1,\vec{n}_1),(p_2,\vec{n}_2))}{\arg\min} E_2[\gamma_\lambda], \tag{4.3}$$

where $\vdash$ means subjecting to the boundary conditions, i.e. γ_λ goes through p_1 and p_2, and $\dot{\gamma}_\lambda \perp \vec{n}_i$ at two ends. As λ varies according to the available boundary data u_0, Equation (4.3) gives a family of (and theoretically infinitely many) elasticas. On the other hand, if we denote this bundle of elasticas by

$$\mathscr{F}' = \{\Gamma'_\lambda : 0 \le \lambda \le 1\},$$

then it is easy to see that $\mathscr{F}'$ is also the minimizer of the following energy for all boundary admissible curve bundles $\mathscr{F} = \{\gamma_\lambda : 0 \le \gamma \le 1\}$:

$$E[\mathscr{F}] = \int_0^1 E_2[\gamma_\lambda]\,\mathrm{d}\lambda, \quad \text{or more generally,} \quad E_w[\mathscr{F}] = \int_0^1 w(\lambda) E_2[\gamma_\lambda]\,\mathrm{d}\lambda, \tag{4.4}$$

with some positive weight function $w(\lambda)$. At this level, two problems can arise

1. Problem 1: two different elastica interpolants Γ'_λ and Γ'_μ, with $\lambda \neq \mu$ can meet inside the inpainting domain D, while the original two isophotes never.

2. Problem 2: generally, it is not guaranteed that the elastica bundle $\mathscr{F}' = \{\Gamma'_\lambda : 0 \le \lambda \le 1\}$ does *wave* the enteire inpainting domain D and leaves no *holes*.

These issues have been taken care in [Masnou and Morel(1998)]. A more convenient alternative approach is to work with the level-set function u_D instead. An ammisible curve boundle $\mathscr{F} = \{\gamma_\lambda\}_\lambda$, which not only satisfies the boundary conditions, but also avoids the above mentioned two problems, is uniquely and fully characterized by an inpainting function u_D that is *tangent* to u_0 along ∂D. However, working with u_D instead of the individual isophotes automatically avoids the above two problems.

The Elastica Inpainting Model

Let $u = u_D$ be an admissible inpainting. Then along any isophote $\gamma_\lambda : u \equiv \lambda$, the curvature of the oriented curve is given by

$$\kappa = \nabla \cdot \vec{n} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|} \right) = \mathbf{H}.$$

Remark 4.10. Given a 2D smooth surface in 3D, if we denote with κ_1 and κ_2 the principal curvatures, respectively the minimum and the maximum curvatures among all curves contained in the orthogonal planes to the surface, we can denote with $\mathbf{H} = \kappa_1 + \kappa_2$ is the *mean curvature* and with $\mathbf{K} = \kappa_1 \kappa_2$ the *Gaussian curvature*. The second fundamental form (or shape form) is

$$A = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}$$

On the other hand, $\mathrm{d}t$ denotes the length element along the normal direction $\vec{n}$ (or along the steepest ascent curve), then we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = |\nabla u|, \quad \text{or} \quad \mathrm{d}\lambda = |\nabla u|\,\mathrm{d}t.$$

Therefore, integated elastica energy (4.4) now passes on to u by

$$
\begin{aligned}
J[u] = E_w[\mathscr{F}] &= \int_0^1 w(\lambda) \int_{\gamma_\lambda : u=\lambda} (a + b\kappa^2)\,\mathrm{d}s\,\mathrm{d}\lambda \\
&= \int_0^1 \int_{\{\gamma_\lambda : u=\lambda\}} w(u)\left(a + b\left(\nabla\cdot\frac{\nabla u}{|\nabla u|}\right)^2\right)|\nabla u|\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_D w(u)\left(a + b\left(\nabla\cdot\frac{\nabla u}{|\nabla u|}\right)^2\right)|\nabla u|\,\mathrm{d}x,
\end{aligned}
$$

since $\mathrm{d}t$ and $\mathrm{d}s$ represents a couple of orthogonal length elements. Now the energy is completely expressed in terms of the inpainting u itself. Notice that this formal derivation is much alike the coarea formula for BV functions. The weights function $w(\lambda)$ can be set to 1. In applications, we can also define it by looking at the histogram $h(\lambda)$ of a given image, where $h(\lambda)$ denotes the frequency of pixels with gray level λ. Since perceptually the regularity of the boundaries (or edges) defining the 2D shapes of the objects is most sensitive to human observers, we may weigh high along such edges whose gray values typically lie near the *valleys* of the histogram. Therefore we may choose the weight function in the form of

$$
w(\lambda) = W(1 - h(\lambda)),
$$

with $W = W(h)$ being a suitable positive and increasing function.

From the moment on, let us consider the functionalized Euler's elastica energy

$$
J_2[u] = \int_D \left(a + b\left(\nabla\cdot\frac{\nabla u}{|\nabla u|}\right)^2\right)|\nabla u|\,\mathrm{d}x, \tag{4.5}
$$

with the conditions that

$$
u|_{\Omega\setminus D} = u_0|_{\Omega\setminus D}, \quad \int_{\partial D} |\nabla u| = 0, \quad |\kappa(p)| < \infty, \text{ a.e. along } \partial D, \tag{4.6}
$$

where a.e. is in the sense of Hausdoorf measure. We have assumed that the original complete image u_0 (typically on a square domain Ω) belongs to $\mathrm{BV}(\Omega)$ and has the property that

$$
\int_{\partial D} |\nabla u_0| = 0,
$$

in the sense of Radon measure $\int |\nabla u_0|$. Under such assumptions, the second boundary condition on u follows naturally and it is called *the feasibility condition* for all low-level inpaintings (i.e. inpaintings which do not depend on global feature recognition or learning). In this way, a low-level inpainting is not expected to *create* a new object, but just *complete* objects based on the hits they left outside the inpainting domain because

$$
\int_{\partial D} |u^+ - u^-|\,\mathrm{d}\mathscr{H}^1 = \int_{\partial D} |\nabla u_0| = 0 \Longrightarrow u^- = u^+ = u_0^+, \text{ a.e. along } \partial D \text{ by } \mathscr{H}^1.
$$

The third condition demands finite curvatures along the inpainting boundary, therefore a sudden turn of isophotes is not permitted along ∂D, and the condition is thus a first order continuity constraint.

Because of the lack of regularity of BV functions it has been introduced the concept of *weak curvature*. Suppose $u \in \mathrm{BV(D)}$, then

$$d_u \nu = \int |\nabla u|$$

is a Radon measure on D. Recall that the TV norm is defined in the distributional sense:

$$\int_D |\nabla u| = \sup_{g \in C_0^1(D,B_1)} \int_D u \, \nabla \cdot g \, dx,$$

where B_1 denotes the unit ball centered at the origin in $\mathbb{R}^2$. Let $\mathrm{supp}(d_u \nu)$ denote the support of the TV measure. Then for any $p \in \mathrm{supp}(d_u \nu)$, on any if its small neighbourhood N_p,

$$d_u \nu(N_p) = \int_{N_p} |\nabla u| > 0.$$

Let ρ be a a fixed radially symmetric non-negative mollifier with compact support and unit total integral, and set (for 2D)

$$\rho_\sigma = \frac{1}{\sigma^2} \rho\left(\frac{x}{\sigma}\right), \text{ and } u_\sigma = \rho_\sigma * u.$$

Definition 4.11. We define the *weak absolute curvature* $\tilde{\kappa}(p)$ of u at p by

$$\tilde{\kappa}(p) = \limsup_{\sigma \to 0} \left| \nabla \cdot \left(\frac{\nabla u_\sigma}{|\nabla u_\sigma|} \right)(p) \right|,$$

where for those σ's which give $|\nabla u_\sigma(p)| = 0$, we define $\nabla \cdot (\nabla u_\sigma / |\nabla u_\sigma|)$ to be ∞.

For any pixel p outside $\mathrm{supp}(d_u \nu)$, we assign 0 to $\tilde{\kappa}(p)$, since u is a.e. a constant near a neighbourhood of p. Thus, the weak absolute curvature is well-defined everywhere for an arbitrary BV function.

Proposition 4.12. *Suppose u is C^2 near p, $p \in \mathrm{supp}(d_u \nu)$, and $\nabla u(p) \neq 0$. Then $\tilde{\kappa}(p) = |\kappa(p)|$.*

Proposition 4.13. *Suppose an oriented curve segment γ is a C^2 submanifold in D. Assume that near a given pixel $p \in \gamma$, to one side of γ, $u = c^+$, and to the other side $u = c^-$, two constant gray values. Then $\tilde{\kappa}(p) = |\kappa(p)|$.*

Proposition 4.14. *Let $u \in \mathrm{BV}(\Omega)$ and $v = au + b$ for some constants a and $b \neq 0$. Then for any $p \in \Omega$, $\tilde{\kappa}_u(p) = \kappa_v(p)$ so, in other words, $\tilde{\kappa}$ is invariant under linear scaling of gray levels.*

With the help of the concept of weak curvature, the functionalized elastica energy (4.5) can be rigorously defined. A BV function u is said to be *admissible* if $\tilde{\kappa} \in \mathrm{L}^2(D, d_u \nu)$. For all such functions, the generalized elastica energy

$$J_2[u] = \int_D (a + b\tilde{\kappa}^2) d_u \nu$$

is well-defined and finite. Toghether with the boundary conditions (4.6), it defines the so called *elastica inpainting model*, which provides a difficult analysis for the existence and uniqueness solution due to the non-convexity of the energy and the involvment of the curvature.

TV inpainting model of Chan and Shen

The TV inpainting model [Chan and Shen(2002)] is an extreme case of the elastica inpainting when one weighs highly against the total variation, i.e. $a/b=\infty$. Thus one is led to the minimization of

$$TV(u)=\int_\Omega |\nabla u|, \quad \text{with the condition} \quad u|_{\Omega\setminus D}=u_0|_{\Omega\setminus D}, \tag{4.7}$$

where D is the enteire (often rectangular) image domain. We shall always assume that Ω is a bounded Lipschitz domain: this is the *noise free TV inpainting model.*

Theorem 4.15 (Existence of a Noise Free TV Inpainting). *Suppose that the original complete image u_0 lies in* BV(Ω)*, and takes gray values between 0 (black) and 1 (white). Then the noise free TV inpainting model, together with the gray value constraint $u\in[0,1]$, has at least one optimal inpainting.*

Proof. Since the original complete image u_0 is admissible (i.e. satisfying the constraint and with finite TV measure), we can always find a minimizing sequence of admissible inpaintings $\{u_h\}_{h\in\mathbb{N}}$ for the model. Then both

$$\int_\Omega |\nabla u_h| \quad \text{and} \quad \int_\Omega |u_h(x)|\,\mathrm{d}x$$

are bounded for all h since Ω is bounded and u_h takes values in the gray scale interval $[0,1]$. By the weak compactness property of BV functions, there is a subsequence, still denoted by $\{u_h\}_{h\in\mathbb{N}}$ for convenience, which strongly converges to some $u_{TV}\in \mathrm{L}^1(\Omega)$ in the L^1 norm. Apparently, u_{TV} still meets the constraints

$$u_{TV}|_{\Omega\setminus D}=u_0|_{\Omega\setminus D} \quad \text{and} \quad u_{TV}(x)\in[0,1].$$

Also, by the L^1 lower semicontinuity property,

$$\int_\Omega |\nabla u_{TV}|\le \liminf_{h\to\infty}\int_\Omega |\nabla u_h| = \min_u \int_\Omega |\nabla u|.$$

Thus u_{TV} must be minimizer. □

Remark 4.16. This model can be modified to deal with noisy images, i.e. $u_0(x)=u_c(x)+n(x)$, where u_c is the *cleaned* image and n is the noise, replacing the constraint (4.7) with:

$$\frac{1}{\mathrm{Area}(\Omega\setminus D)}\int_{\Omega\setminus D}(u-u_0)^2=\sigma^2, \tag{4.8}$$

where σ^2 is the variation of the noise, which can be estimated from $u_0|_{\Omega\setminus D}$ by statistical estimators. We can however assume that $n(x)$ is Gaussian with mean 0.

Theorem 4.17 (Existence of a TV Inpainting for a Noisy Image). *Given an image observation u_0 on $\Omega\setminus D$, assume that there exists at least one image u_c (i.e. the original* clean *image) on Ω, which belongs to* BV(Ω) *and meets the denoising constraint* (4.8) *and gray scale constraint $u_c\in[0,1]$. Then there exists at least one optimal TV inpainting on Ω, which does inpainting inside and noise cleaning outside.*

Remark 4.18. The solutions to both TV and elastica inpaintings can be non-unique: this is an intrinsic inpainting problem.

4.3 A variational approach for exemplar-based inpainting

Referred to Figure 12, we introduce the main aspects of the very recent work on inpainting problem by [Arias *et al.*(2011)]. We denote the images as functions $u : \Omega \to \mathbb{R}$, where Ω denotes the image domain, usually a rectangle in $\mathbb{R}^n$: the points in Ω are pixels. Pixel positions are denoted by x, y or h, the latter for positions inside the patch. A patch of u centred at x is denoted by $p_u(x) = p_u(x, \cdot) : \Omega_p \to \mathbb{R}$ where Ω_p is a rectangle centred at 0. The patch is defined by $p_u(x,h) = u(x+h)$, with $h \in \Omega_p$. Let $O \subset \Omega$ be the hole or inpainting domain, and $O^c = \Omega \setminus O$. We assume that O is an open set with Lipschitz boundary. We still denote by u the part of the image u inside the hole, while $\overline{u}$ is the part of u in O^c : $\overline{u} = u|_{O^c}$. We take $\widetilde{O}$ as the set of centres of patches that intersect the hole, i.e. $\widetilde{O} = O + \Omega_p = \{x \in \Omega : (x + \Omega_p) \cap O \neq \emptyset\}$. In doing so, patches $p_{\hat{u}}(y)$ centred at points $y \in \widetilde{O}^c$ are contained in O^c and the Euler equation for the minimizer of the proposed functional is simplified. We assume that $\widetilde{O} + \Omega_p \subset \Omega$, i.e. every pixel in $\widetilde{O}$ supports a patch centred on it and contained in Ω. Analogously, we will also shrink $\widetilde{O}^c$ to have $\widetilde{O}^c + \Omega_p \subset \Omega$.

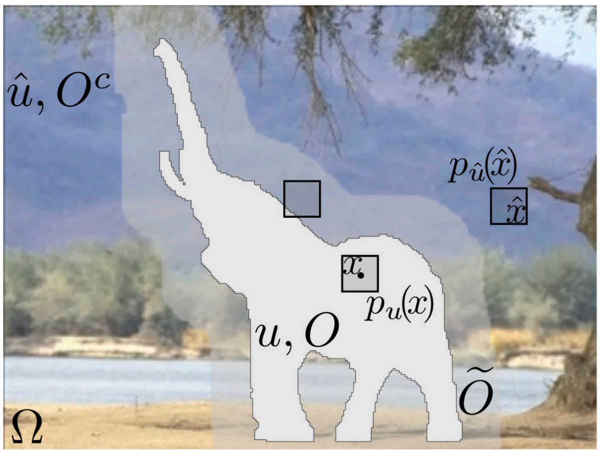


Figure 12: Notations used in inpainting problem.

Energy Formulation

The proposed energy formulation contains two terms, one of them is inspired by the following functional

$$F_w(u) = \int_O \int_{O^c} w(x,y)(u(x) - \hat{u}(y))^2 \, dy \, dx, \tag{4.9}$$

where $w : O \times O^c \to \mathbb{R}^+$ is a (probabilistic density) weight function that measures the similarity between patches centered in the inpainting domain and in its complement, while the second one allows us to compute the weigths given the image. If $w(x,y)$ are known, the minimum of Equation (4.9) should have a low pixel error $(u(x)-\widehat{u}(y))^2$ whenever the similarity weight is high, driving the information transfers from known to unknown pixels. The complete proposed functional is

$$\mathscr{E}_{\varepsilon,T}(u,w) = \mathscr{U}_\varepsilon(u,w) - T\int_{\widetilde{O}} \mathscr{H}(w(x,\cdot))\,\mathrm{d}x, \tag{4.10}$$
$$\text{subject to } \int_{\widetilde{O}^c} w(x,y) = 1,$$

where

$$\mathscr{U}_\varepsilon(u,w) = \int_{\widetilde{O}}\int_{\widetilde{O}^c} w(x,y)\varepsilon\,(p_u(x)-p_{\widehat{u}}(y))\,\mathrm{d}y\,\mathrm{d}x, \tag{4.11}$$

$\varepsilon(\,\cdot\,)$ is an error function for image patches (such as the squared L^2 norm), and

$$\mathscr{H}(w(x,\cdot)) = -\int_{\widetilde{O}^c} w(x,y)\log(w(x,y))\,\mathrm{d}y$$

is the entropy of the probability $w(x,\cdot)$. Minimizing Equation (4.11) with respect to the image u will force the patch $p_u(x)$ to be similar to $p_u(y)$ whenever $w(x,y)$ is high. Moreover, for a given completion u, and for each $x \in \widetilde{O}$, the optimum weights minimize the mean patch error for $p_u(x)$ given by

$$\int_{\widetilde{O}^c} w(x,y)\varepsilon(p_u(x)-p_{\widehat{u}}(y))\,\mathrm{d}y,$$

while maximizing the entropy. This can be related to the *principle of maximum entropy* [Jaynes(1957)], widely used for inference of probability distributions. According to it, the best representation for a distribution, given a set of samples, is that one maximizing the entropy, i.e. the distribution which makes less assumptions about the process. Taking ε as the squared L^2-norm of the patch, then the resulting weights are given by

$$w(x,y) \propto \exp\left(-\frac{1}{T}\|p_u(x)-p_{\widehat{u}}(y)\|^2\right),$$

where T controls the trade-off between both terms and is also the selectivity parameter of the Gaussian weights. Note that restricting $w(x,\cdot)$ to be a probability, trivial minima of ε with $w(x,y)=0$ everywhere are discarded.

The patch error function ε

Patches are functions defined on Ω_p and, if $\mathbb{P}$ denotes a suitable space of patches, the error function $\varepsilon : \mathbb{P} \to \mathbb{R}^+$ is defined as the weighted sum of pixel-wise errors $e : \mathbb{R} \to \mathbb{R}^+$:

$$\varepsilon(p_u(x)-p_{\widehat{u}}(y)) = g * e(u(x+\cdot)-\widehat{u}(y+\cdot)).$$

Here, $g : \mathbb{R}^n \to \mathbb{R}^+$ denotes a suitable *intra-patch* kernel function (with the highly desiderable property to have a compact support).

Patch non-local means. If we use $e(r)=|r|^2$, then $\mathbb{P}\equiv \mathrm{L}^2(\Omega_p)$ and

$$\varepsilon(p_u(x)-p_{\widehat{u}}(y))=||p_u(x)-p_{\widehat{u}}(y)||_g^2=g*|u(x+\cdot)-\widehat{u}(y+\cdot)|^2.$$

Patch non-local medians. If we use $e(r)=|r|$, then $\mathbb{P}\equiv \mathrm{L}^1(\Omega_p)$ and

$$\varepsilon(p_u(x)-p_{\widehat{u}}(y))=||p_u(x)-p_{\widehat{u}}(y)||_g=g*|u(x+\cdot)-\widehat{u}(y+\cdot)|.$$

Patch non-local Poisson. If we use $\mathbb{P}\equiv \mathrm{W}^{1,2}(\Omega_p)$, then

$$\varepsilon(p_u(x)-p_{\widehat{u}}(y))=||p_u(x)-p_{\widehat{u}}(y)||_{\nabla,g}^2=g*|\nabla u(x+\cdot)-\nabla \widehat{u}(y+\cdot)|^2.$$

Patch non-local gradient medians. If we use $\mathbb{P}\equiv \mathrm{BV}(\Omega_\mathrm{p})$, then

$$\varepsilon(p_u(x)-p_{\widehat{u}}(y))=||p_u(x)-p_{\widehat{u}}(y)||_{\nabla,g}=g*|\nabla u(x+\cdot)-\nabla \widehat{u}(y+\cdot)|.$$

The information provided by the gradient of the image allow to determine not only the patch similarity but also the image synthesis. An energy for RGB images can be obtained by defining a patch error function for RGB patches as the sum of the error functions of the three scalar components:

$$\varepsilon(p_u(x)-p_{\widehat{u}}(y))=\sum_{i=1}^{3}\varepsilon(p_{u_i}(x)-p_{\widehat{u}_i}(y)).$$

The Euler-Lagrage equation

Let us compute the Euler-Lagrange equations of $\mathscr{E}_{\varepsilon,T}$ with respect to both weights and image. Fixed u and minimizing Equation (4.10) with respect to w we obtain, from the Euler-Lagrange equation $\delta_w\mathscr{E}_{\varepsilon,T}(u,w)=0$,

$$w_{\varepsilon,T}(u)(x,y)=\frac{1}{Z_{\varepsilon,T}(u)(x)}\exp\Big(-\frac{1}{T}\varepsilon(p_u(x)-p_{\widehat{u}}(y))\Big),$$

with the normalizing factor (to obtain a probability)

$$Z_{\varepsilon,T}(u)(x)=\int_{\widetilde{O}^c}\exp\Big(-\frac{1}{T}\varepsilon(p_u(x)-p_{\widehat{u}}(y))\Big)\mathrm{d}y.$$

The weight function $w(x,y)$ measures the similarity between the patches centred at $x\in\widetilde{O}$ and $y\in\widetilde{O}^c$.

For computing the Euler-Lagrange equation with respect to the image, we will consider the energies corresponding to the patch NL-Means and NL-Poisson patch error functions

Patch NL-means The resulting Euler-Lagrange equation is the following

$$u(z)=\frac{1}{k(w)(z)}\int_{\mathbb{R}^n}g*w(z-\cdot,z'-\cdot)\widehat{u}(z')\mathrm{d}z',\quad z\in O,$$

where $k(w)(z)=\int_{\mathbb{R}^n}g*w(z-\cdot,z'-\cdot)\mathrm{d}z'=1$, assuming both the weights and g are normalized. Thus, optimal u are given by a non-local average of the known pixels. The weights in the average are obtained by convolving the Gaussian similarity weights with the patch kernel g.

Patch NL-Poisson In this case we have that u is a solution of the Poisson equation:

$$\begin{cases} \Delta u(z) = \operatorname{div} \mathbf{v}(w)(z), & z \in O, \\ u = \widehat{u}, & \text{in } \partial O, \end{cases} \tag{4.12}$$

where $\mathbf{v}(w)(z) = \int_{\mathbb{R}^n} g * w(z - \cdot, z' - \cdot) \nabla \widehat{u}(z') \mathrm{d}z'$. The solutions of this Poisson equation are minimizers of the functional $\int_{\widetilde{O}} \|\nabla u(z) - \mathbf{v}(w)(z)\|_2^2 \mathrm{d}z$. Therefore, u is computed as the image with the closest gradient, in the L^2 sense, to a *guiding vector field* $\mathbf{v}(w)(z)$ computed as a non-local average of the image gradients in the known portion of the image.

Getting a correpondence

It is possible to get formally a correspondence map by taking the limit $T \to 0$. The resulting energy is dominated by the image term and can be written as

$$E(u, w) \simeq \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x, \widehat{x}) \varepsilon(p_u(x) - p_{\widehat{u}}(\widehat{x})) \mathrm{d}\widehat{x} \, \mathrm{d}x,$$

rewritten as

$$E(u, \varphi) \simeq \int_{\widetilde{O}} \varepsilon(p_u(x) - p_{\widehat{u}}(\varphi(x))) \mathrm{d}x.$$

So, the weights $w(x, \cdot)$ can be written as Dirac's delta function on a point $\varphi(x)$ which is a nearest neighbour of the patch $p_u(x)$ with respect to the patch error function, i.e. $w(x, \widehat{x}) = \delta(\widehat{x} - \varphi(x))$.

Existence of minima

Patch NL-means model

We assume that Ω is a rectangle in $\mathbb{R}^n$ and $\widehat{u} : O^c \to \mathbb{R}$ with $\widehat{u} \in \mathrm{L}^\infty(O^c)$. We assume that $u : \Omega \to \mathbb{R}$ is such that $u|_{O^c} = \widehat{u}$. We also assume that u is extended by symmetry and then by periodicity to $\mathbb{R}^n$. We consider the patch NL-Means model

$$\mathscr{E}_{2,T}(u, w) = \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x, y) \|p_u(x) - p_{\widehat{u}}(y)\|_g^2 + T \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x, y) \log w(x, y) \, \mathrm{d}y \, \mathrm{d}x.$$

Obviously, $\mathscr{E}_{2,T}(u, w) = +\infty$ in case that the second integral is not defined. Let

$$\mathscr{W} = \left\{ w \in \mathrm{L}^1(\widetilde{O} \times \widetilde{O}^c) : \int_{\widetilde{O}^c} w(x, y) \mathrm{d}y = 1 \text{ a.e. } x \in \widetilde{O} \right\}.$$

Our purpose is to prove the following result stating the existence of minima of

$$\min_{(u,w) \in \mathscr{A}_2} \mathscr{E}_{2,T}(u, w), \tag{4.13}$$

where $\mathscr{A}$ is the admissible class of functions

$$\mathscr{A}_2 = \left\{(u,w) : u \in \mathrm{L}^\infty(\Omega), u = \widehat{u} \text{ in } O^c, w \in \mathscr{W}\right\}.$$

Let $C_c(\mathbb{R}^n)$ be the set if continuous functions with compact support in $\mathbb{R}^n$ and $C_c(\mathbb{R}^n)^+$ the set of nonnegative functions in $C_c(\mathbb{R}^n)$. Let Q an open set and $W^{1,p}(Q)$, $1 \leq p \leq \infty$ is the space of functions $v \in \mathrm{L}^p(Q)$ such that $\nabla v \in \mathrm{L}^p(Q)^n$. By $W^{1,p}(Q)^+$ we denote the set of nonnegative functions in $W^{1,p}(Q)$. We denote $W^{2,p}(Q)$ (resp. by $W^{2,p}_{\mathrm{loc}}(Q)$), $1 \leq p \leq \infty$, the space of functions $v \in \mathrm{L}^p(Q)$ such that $\nabla v \in \mathrm{L}^p(Q)^n$ and $D^2 v \in \mathrm{L}^p(Q)^{n\times n}$ (resp. the functions $v \in W^{2,p}(Q')$) for any subdomain Q' included in a compact set of Q. Let us assume in the rest that $g \in \mathrm{L}^1(\mathbb{R}^n)^+$ and $\int_{\mathbb{R}^n} g(h)\,\mathrm{d}h = 1$.

Proposition 4.19. *Assume that $g \in C_c(\mathbb{R}^n)^+$ has support contained in Ω_p, $\nabla g \in \mathrm{L}^1(\mathbb{R}^n)$ and $\widehat{u} \in$ $\mathrm{BV}(\mathrm{O^c}) \cap \mathrm{L}^\infty(O^c)$.*

1. *If $(u_n, w_n) \in \mathscr{A}_2$ is a minimizing sequence for $\mathscr{E}_{2,T}$ such that u_n is uniformly bounded, then we may extract a subsequence converging to a minimum of $\mathscr{E}_{2,T}$.*
2. *There exist a minimum $(u,w) \in \mathscr{A}_2$ of $\mathscr{E}_{2,T}$. For any minimum $(u,w) \in \mathscr{A}_2$ we have that $u \in W^{1,\infty}(O)$ and $w \in W^{1,\infty}(\widetilde{O} \times \widetilde{O}^c)$.*

In other words, there are smooth minima and smooth probability distributions representing the fuzzy correspondences between $\widetilde{O}$ and $\widetilde{O}^c$. To prove Proposition 4.19, we need the following lemma:

Lemma 4.20. *Assume that $g \in C_c(\mathbb{R}^n)^+$ has support contained in Ω_p, $\nabla g \in \mathrm{L}^1(\mathbb{R}^n)$ and $\widehat{u} \in \mathrm{BV}(\mathrm{O^c}) \cap$ $\mathrm{L}^\infty(O^c)$. Assume that $u \in \mathrm{L}^\infty(\widetilde{O} + \Omega_p)$. Then the functions*

$$\nabla_x g * (u(x+\cdot) - \widehat{u}(y+\cdot))^2 \quad \textit{and} \quad \nabla_y g * (u(x+\cdot) - \widehat{u}(y+\cdot))^2$$

are uniformly bounded in $\widetilde{O} \times \widetilde{O}^c$ by a constant that depends on $||\nabla g||_{L^1}$, $||u||_\infty$ and $||\widehat{u}||_\infty$.

Remark 4.21. The assumption $\widehat{u} \in \mathrm{BV}(\mathrm{O^c})$ forces to consider the assumption that $\widehat{u} \in \mathrm{L}^\infty(O^c)$.

Proof of Proposition 4.19. Let us prove the two statements:

1. Let $(u_n, w_n) \in \mathscr{A}_2$ be a minimizing sequence of Equation (4.13) such that $\{u_n\}_n$ is uniformly bounded. Since Ω is a bounded domain we have that the following integral is bounded:

   $$\int_{\widetilde{O}} \int_{\widetilde{O}^c} \chi_{\{w_n > 1\}} w_n(x,y) \log w_n(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

   Hence $w_n(1+\log^+ w_n)$ is bounded in $\mathrm{L}^1(\widetilde{O} \times \widetilde{O}^c)$, i.e. w_n is bounded in $LLog^+\mathrm{L}(\widetilde{O} \times \widetilde{O}^c)$, called Birnbaum-Orlicz space (the space of functions such that $\int_{\mathbb{R}^n} |u| \log^+ |u| < \infty$, this generalize the $\mathrm{L^p}$ spaces). Then the sequence w_n is relatively weakly compact in L^1 and modulo a subsequence we may assume that w_n weakly converges in $L^1(\widetilde{O} \times \widetilde{O}^c)$ to some $w \in \mathscr{W}$. By Lemma 4.20, the functions

$$\nabla_x \int_{\mathbb{R}^n} g(h)(u_n(x+h) - \widehat{u}(y+h))^2 \, dh \quad \text{and} \quad \nabla_y \int_{\mathbb{R}^n} g(h)(u_n(x+h) - \widehat{u}(y+h))^2 \, dh$$

are bounded in $L^\infty(\widetilde{O} \times \widetilde{O}^c)$. Thus, modulo the extraction of a subsequence, we may assume that $u_n \to u$ weakly in all L^p, $1 \le p < \infty$ and $g * (u_n(x+\cdot) - \widehat{u}(y+\cdot))^2$ converges strongly in all L^p spaces and also in the dual of $LLog^+L$ to some function W. Then, by passing to the limit as $n \to \infty$ we have

$$\int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) W(x,y) \, dy \, dx + T \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) \log w(x,y) \, dy \, dx \le \liminf_n \varepsilon_{2,T}(u_n, w_n).$$

Taking test functions $\psi(x,y)$, integrating in $\widetilde{O} \times \widetilde{O}^c$ and using the convexity of the square function, we have

$$\int_{\mathbb{R}^n} g(h)(u(x+h) - \widehat{u}(y+h))^2 \, dh \le W(x,y).$$

Thus,

$$\mathscr{E}_{2,T}(u,w) \le \liminf_n \mathscr{E}_{2,T}(u_n, w_n).$$

2. See [Arias *et al.*(2012)] for details. □

Patch NL-Poisson model

We consider the model

$$\mathscr{E}_{\nabla,T}(u,w) = \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) \|p_u(x) - p_{\widehat{u}}\|^2_{g,\nabla} \, dy \, dx + T \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) \log w(x,y) \, dy,$$

where,

$$\|p\|^2_{g,\nabla} = \int_{\mathbb{R}^n} g(h) \|\nabla p(h)\|_2^2 \, dh, \quad p \in \mathscr{P}.$$

Recall that we assume that $u|_{O^c} = \widehat{u}$. Let

$$\mathscr{A}_\nabla = \left\{ (u,w) \in \mathscr{A}_2 : u \in W^{1,2}(O), u|_{\partial O} = \widehat{u}|_{\partial O^c} \right\}.$$

Our purpose is to prove the following result stating the existence of minima of

$$\min_{(u,w) \in \mathscr{A}_\nabla} \mathscr{E}_{\nabla,T}(u,w). \tag{4.14}$$

Proposition 4.22. *Assume that $\widehat{u} \in W^{2,2}(O^c) \cap L^\infty(O^c)$ and $g \in W^{1,\infty}(\mathbb{R}^n)^+$ has compact support in Ω_p. There exists a solution of the variational problem* (4.14)*. Moreover for any solution $(u,w) \in \mathscr{A}_\nabla$ we have $u \in W^{1,2}(O) \cap W^{2,p}_{\mathrm{loc}}(O) \cap L^\infty(O)$ for all $p \in [1,\infty]$ and $w \in W^{1,\infty}(\widetilde{O} \times \widetilde{O}^c)$.*

Lemma 4.23. *Assume that $\widehat{u} \in \mathrm{W}^{1,2}(O^c)$ and $g \in \mathrm{L}^\infty(\mathbb{R}^n)^+$ has compact support on Ω_p. Let $(u,w) \in \mathscr{A}_\nabla$. Assume that $\varepsilon_{\nabla,T}(u,w) \leq C$. Then*

$$||u||_{\mathrm{W}^{1,2}(O)} \leq C'\left(C, ||\nabla \widehat{u}||_{\mathrm{L}^2(O^c)}\right),$$

where $C' = C'\left(C, ||\nabla \widehat{u}||_{\mathrm{L}^2(O^c)}\right)$ denotes a constant that depends on its arguments.

Lemma 4.24. *Assume that $\widehat{u} \in \mathrm{W}^{2,2}(O^c)$, $u \in \mathrm{W}^{1,2}(O)$, $u|_{\partial O} = \widehat{u}|_{\partial O^c}$ and $g \in \mathrm{W}^{1,\infty}(\mathbb{R}^n)^+$ has compact support on Ω_p. Then*

$$\nabla_x \int_{\mathbb{R}^n} g(h)|\nabla_x u(x+h) - \nabla_y \widehat{u}(y+h)|^2 \,\mathrm{d}h \quad \textit{and} \quad \nabla_y \int_{\mathbb{R}^n} g(h)|\nabla_x u(x+h) - \nabla_y \widehat{u}(y+h)|^2 \,\mathrm{d}h$$

are bounded in $\mathrm{L}^\infty(\widetilde{O} \times \widetilde{O}^c)$ with a bound depending on $||\widehat{u}||_{\mathrm{W}^{2,2}(O^c)}$, $||g||_{\mathrm{W}^{1,\infty}}$ and $||\nabla u||_{\mathrm{L}^2(O)}$.

Proof of Proposition 4.22. Let us prove the two statements:

1. Let (u_n, w_n) be a minimizing sequence of Equation (4.14). Since Ω is a bounded domain we have that
$$\int_{\widetilde{O}} \int_{\widetilde{O}^c} \chi_{\{w_n > 1\}} w_n(x,y) \log w_n(x,y) \,\mathrm{d}y \,\mathrm{d}x$$
is bounded. Hence $w_n(1 + \log^+ w_n)$ is bounded in $\mathrm{L}^1(\widetilde{O} \times \widetilde{O}^c)$. Then the sequence w_n is relatively weakly compact in L^1 and modulo a subsequence we may assume that w_n weakly converges in $\mathrm{L}^1(\widetilde{O} \times \widetilde{O}^c)$ to some $w \in \mathscr{W}$. By Lemma 4.23, we have that u_n is uniformly bounded in $\mathrm{W}^{1,2}(\widetilde{O})$. By Lemma 4.24, we have that
$$\nabla_x \int_{\mathbb{R}^n} g(h)|\nabla u(x+h) - \nabla u(y+h)|^2 \,\mathrm{d}h \quad \text{and} \quad \nabla_y \int_{\mathbb{R}^n} g(h)|\nabla u(x+h) - \nabla u(y+h)|^2 \,\mathrm{d}h$$
are uniformly bounded in $\mathrm{L}^\infty(\widetilde{O} \times \widetilde{O}^c)$. Thus, modulo the extraction of a subsequence, we may assume that $u_n \to u$ a.e. and in $\mathrm{L}^2(\widetilde{O})$, $\nabla u_n \to \nabla u$ weakly in $\mathrm{L}^2(\widetilde{O} + \Omega_p)$ and $g * (\nabla_x u_n(x + \cdot) - \nabla_y \widehat{u}_n(y + \cdot))^2$ converges strongly in all L^p spaces, $1 \leq p < \infty$, and also in the dual of $LLog^+\mathrm{L}$ to some function W. Then by passing to the limit as $n \to \infty$, we have
$$\int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) W(x,y) \,\mathrm{d}y \,\mathrm{d}x + T \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) \log w(x,y) \,\mathrm{d}y \,\mathrm{d}x \leq \liminf_n \varepsilon_{\nabla,T}(u_n, w_n).$$
Taking test functions $\psi(x,y)$, integrating in $\widetilde{O} \times \widetilde{O}^c$ and using the convexity of the square function, we have
$$\int_{\mathbb{R}^n} g(h)(\nabla_x u(x+h) - \nabla_y \widehat{u}(y+h))^2 \,\mathrm{d}h \leq W(x,y).$$
Thus
$$\mathscr{E}_{\nabla,T}(u,w) \leq \liminf_n \mathscr{E}_{\nabla,T}(u_n, w_n).$$
2. See [Arias *et al.*(2012)] for details. □

Existence of optimal correspondence maps

Definition 4.25 (Measurable measure-valued map). Let $\mathscr{X} \subseteq \mathbb{R}^n$, $\mathscr{Y} \subseteq \mathbb{R}^m$ be open sets, μ be a positive Radon measure in $\mathscr{X}$ and $x \to \nu_x$ be a function that assigns to each $x \in \mathscr{X}$ a Radon measure ν_x on $\mathscr{Y}$. We say that the map is μ-measurable if $x \to \nu_x(B)$ is μ-measurable for any Borel set $B \in \mathscr{Y}$.

By the disintegration Theorem, if ν is a Radon measure in $\mathscr{X} \times \mathscr{Y}$ such that $\nu(\mathscr{X} \times \mathscr{Y}) < \infty$ for any compact set $K \subseteq \mathscr{X}$ and $\mu = \pi_\# \nu$ (i.e. $\mu(B) = \nu(B \times Y)$ for any Borel set $B \subseteq X$, where $\pi : \mathscr{X} \times \mathscr{Y} \to \mathscr{X}$ is the projection on the first factor), then there exist a measurable measure-valued map $x \to \nu_x$ such that $\nu_x(\mathscr{Y}) = 1$ μ-a.e. in $\mathscr{X}$ and for any $\psi \in \mathrm{L}^1(\mathscr{X} \times \mathscr{Y}, \nu)$ we have,

$$\begin{gathered} \psi(x, \cdot) \in \mathrm{L}^1(\mathscr{Y}, \nu_x), \quad \text{for } \mu\text{-a.e. } x \in \mathscr{X}, \\ x \to \int_{\mathscr{Y}} \psi(x,y)\, \mathrm{d}\nu_x(y) \in \mathrm{L}^1(\mathscr{X}, \mu), \\ \int_{\mathscr{X} \times \mathscr{Y}} \psi(x,y)\, \mathrm{d}\nu(x,y) = \int_{\mathscr{X}} \int_{\mathscr{Y}} \psi(x,y)\, \mathrm{d}\nu_x(y)\, \mathrm{d}\mu(x). \end{gathered}$$

Let us consider $\mathscr{M}\mathscr{P}$ the set of measurable measure valued maps $\nu \geq 0$ in $\widetilde{\mathrm{O}} \times \mathrm{cl}(\widetilde{\mathrm{O}}^c)$ such that $\pi_\# \nu = \mathscr{L}^n|_{\widetilde{\mathrm{O}}}$ denotes the Lebesgue measure restricted to $\widetilde{\mathrm{O}}$. We assume that $g \in C_c(\mathbb{R}^n)$ has support contained in Ω_p, $\nabla g \in \mathrm{L}^1(\mathbb{R}^n)$ and $\widehat{u} \in \mathrm{BV}(\mathrm{O^c}) \cap \mathrm{L}^\infty(O^c)$. Let

$$\mathscr{A}_{2,0} = \{(u,\nu) : u \in \mathrm{L}^\infty(\Omega),\ u = \widehat{u} \text{ in } \mathrm{O}^c,\ \nu \in \mathscr{M}\mathscr{P}\}.$$

For $(u,\nu) \in \mathscr{A}_{2,0}$, define

$$\mathscr{E}_{2,0}(u,\nu) = \int_{\widetilde{\mathrm{O}}} \int_{\widetilde{\mathrm{O}}^c} g * (u(x + \cdot) - \widehat{u}(y + \cdot))^2\, \mathrm{d}\nu(x,y).$$

By Lemma 4.20, the above integral is well defined.

Theorem 4.26. *There exists a minimum $(u,\nu) \in \mathscr{A}_{2,0}$ of $\mathscr{E}_{2,0}$.*

Let $\varphi : \widetilde{\mathrm{O}} \to \widetilde{\mathrm{O}}^c$ be a measurable map. Then $x \in \widetilde{\mathrm{O}} \to \nu_x = \delta_{\varphi(x)}(y)$ is measurable. Similarity if the map $x \in \widetilde{\mathrm{O}} \to \nu_x = \delta_{\varphi(x)}(y)$ is measurable then φ is measurable. Let us denote by ν^φ the measure determined by φ.

Proposition 4.27. *There exists a minimum $(u^*,\nu^*) \in \mathscr{A}_{2,0}$ of $\mathscr{E}_{2,0}$ such that $\nu^* = \nu^\varphi$ where $\varphi : \widetilde{\mathrm{O}} \to \mathrm{cl}(\widetilde{\mathrm{O}}^c)$ is a measurable map.*

The next proposition shows the relation between the patch NL-Means functional for $T > 0$ and $\mathscr{E}_{2,0}$.

Proposition 4.28. *The energies $\mathscr{E}_{2,T}$ Γ-converge to the energy $\mathscr{E}_{2,0}$. In particular, the minima of $\mathscr{E}_{2,T}$ converge to minima of $\mathscr{E}_{2,0}$.*

Remark 4.29. For the NL-Poisson model, the limit energy is now

$$\mathscr{E}_{\nabla,0}(u,\nu) = \int_{\widetilde{O}} \int_{\widetilde{O}^c} g * (\nabla u(x+\cdot) - \nabla \widehat{u}(y+\cdot))^2 \, d\nu(x,y).$$

In this case, we assume that $\widehat{u} \in W^{2,2}(O^c)$ and $g \in W^{1,\infty}(\mathbb{R}^n)$ has compact support in Ω_p and we use Lemmas 4.23 and 4.24.

Alternating optimization

NL-means algorithm

Proposition 4.30. *The iterated optimization algorithm converges (modulo a subsequence) to a critical point of $\mathscr{E}_{2,T}$. The solution obtained (u^*, w^*) satisfies the regularity properties stated in Proposition 4.19, that is $u^* \in W^{1,\infty}(O)$ and $w^* \in W^{1,\infty}(\widetilde{O} \times \widetilde{O}^c)$.*

Algorithm 1 Alternating optimization for NL-means model

Input: u^0 with $||u^0||_\infty \leq ||\widehat{u}||_\infty$.
1: **for** each $k \in \mathbb{N}$ **do**
2: $w^{k+1} = \arg\min_{w \in \mathscr{W}} \mathscr{E}_{2,T}(u^k, w)$,
3: $u^{k+1} = \arg\min_u \mathscr{E}_{2,T}(u, w^{k+1})$.
4: **end for**

NL-Poisson algorithm

Proposition 4.31. *The iterated optimization algorithm converges (modulo a subsequence) to a critical point $(u^*, w^*) \in \mathscr{A}_\nabla$ of the energy $\mathscr{E}_{\nabla,T}(u,w)$. The solution obtained has the smoothness described in Proposition 4.22, i.e. $u \in W^{1,2}(O) \cap W^{2,p}_{loc}(O) \subset L^\infty(O)$ for any $p \in [1,\infty)$ and $w \in W^{1,\infty}(\widetilde{O} \times \widetilde{O}^c)$.*

Algorithm 2 Alternating optimization for NL-Poisson model

Input: u^0 with $||u^0||_\infty \leq ||\widehat{u}||_\infty$.
1: **for** each $k \in \mathbb{N}$ **do**
2: $w^{k+1} = \arg\min_{w \in \mathscr{W}} \mathscr{E}_{\nabla,T}(u^k, w)$,
3: $u^{k+1} = \arg\min_{u \in W^{1,2},\, u|_{\partial O^c} = \widehat{u}|_{\partial O^c}} \mathscr{E}_{\nabla,T}(u, w^{k+1})$.
4: **end for**

Remarks and visual results

Exemplar-based inpainting is based on patch similarity and this comparison is performed by the *Patchmatch* algorithm [Barnes *et al.*(2009)]. Patch search for similarities is a common problem in many applications and the computation of the nearest neighbour is the most time consuming step: Patchmatch performs the search among all patches in the domain very fast (0.72s for two 399×358 images). This algorithm was introduced for computing nearest neighbours between two images (e.g. similar patches with respect to L^2 metric). The justification of the convergence is based on a simple probabilistic argument:

1. randomly finding a match for a particular pixel is rare and, if M is the image size, then the probability to choose the best match is $\mathbb{P} = 1/M$;
2. matching for at least one (in the image) is not rare at all: $1 - 1/M$ is the probability to make the wrong choice
$$\mathbb{P} = 1 - \left(1 - \frac{1}{M}\right)^M \approx 1 - \frac{1}{e} \quad \text{for large } M;$$
3. finding an approximate match (i.e. the offset is in a C-neighbourhood of the exact offset) that will be refined in the next iterations has even more chance:
$$\mathbb{P} = 1 - \left(1 - \frac{C}{M}\right) \approx 1 - \exp(-C);$$
4. propagation takes care of the rest.

(a) Input. (b) Output. (c) Patch-pasting.

Figure 13: Results from Figure 12.

Another issue of Exemplar-based inpainting methods is the critical dependence with the size of the patch. Furthermore, when the inpainting domain is large in comparison with the patch, the proposed energies have many local minima, and not all of them are *good* inpaintings. It is a common practice in literature to incorporate a multiscale scheme, applying sequentially the inpainting method on a Gaussian image pyramid, starting at the coarsest scale. The result at each scale is upsampled and used as initialization for the next finer scale while the patch size is constant through scales: this corresponds to minimize a sequence of energies with decreasing patch size without subsampling the image. Among all the experiments in [Arias *et al.*(2011)], we show two meaningful examples compared with a method in [Kawai *et al.*(2009)], based on considering brightness change and spatial locality of texture pattern: this allows linear brightness change of the texture pattern. Results in Figure 14.

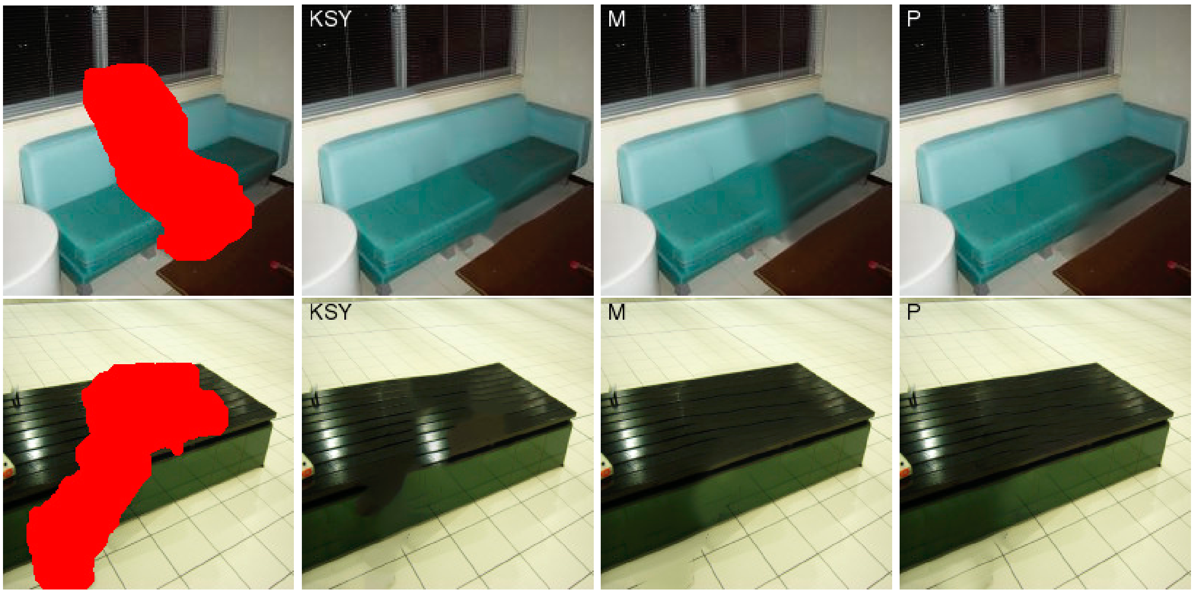


Figure 14: KSY from [Kawai *et al.*(2009)] while M and P stand for patch NL-Means, and NL-Poisson.

Remark 4.32. The NL-Poisson method fits very well the transition of brightness while NL-Means is useful in blending situations. Despite of this, one of the main issue in this research field is to solve the inpainting problem for a domain in which the perspective and the illumination change simultaneously: the copy-past method doesn't work well in this context.

CHAPTER 5

A Drift Diffusion approach for Shadow Removal

Among all variational methods proposed by [Arias *et al.*(2011)] for solving the inpainting problem, we have seen that *Non-local Poisson method* models very well the situations where gradients of illumination are highly desirable. This behaviour is strictly connected with the diffusion-transport Equation (4.12) to be solved. This equation is also called *drift-diffusion* equation, derived from the *Fokker-Plank* probabilistic equation which describes the time evolution of the probability density function of the velocity of a particle under drag and random forces' influence, as in Brownian motion:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}[D(x,t)u(x,t)] - \frac{\partial}{\partial x}[\mu(x,t)u(x,t)],$$

with drift $\mu(X_t, t)$ and diffusion coefficient $D(X_t, t)$.

In this chapter, we present the main results of [Weickert *et al.*(2013), Vogel *et al.*(2013)], where special applications of drift-diffusion equation are shown: in particular, our focus is in recovering the information underlying shadow areas. We tested the proposed formulation working out also some alternative approaches for solving the PDE, in order to speed up the computation time. Drift-diffusion equation is sometimes presented as an osmosis process, in the sense reported by Prof. Weickert:

> *Osmosis is a transport phenomenon that is omnipresent in nature. It differs from diffusion by the fact that is allows nonconstant steady states. [...] Osmosis describes transport through a semipermeable membrane in such a way that in its steady state, the liquid concentrations on both sides of the membrane can differ. Osmosis is the primary mechanism for transporting water in and out of cells, and it has many applications in medicine and engineering. It can be seen as the nonsymmetric counterpart of diffusion. Since diffusion can only model symmetric transport processes, it leads to flat steady states. [...] In contrast to osmosis in natural systems we do not need two different phases (water and salt) and a membrane that is only permeable for one of them: We can obtain nonconstant steady states within a single phase that represents the grey value.*

Definition 5.1 (Steady State). In systems theory, a system is in *steady state* for a property p if the partial derivative of p with respect to time is zero, i.e.

$$\frac{\partial p}{\partial t} = 0$$

Steady state is a more general situation than dynamic equilibrium. If a system is in steady state, then the recently observed behaviour of the system will continue into the future. Typically, the state between the initial data and the steady state are called *transient state*.

5.1 The continuous model

For simplicity, in Image Processing a grayscale image is considered as positive function u on a rectangular domain $\Omega \subset \mathbb{R}^2$:

$$u : \Omega \to \mathbb{R}^+, \text{ with boundary } \partial\Omega.$$

Definition 5.2 (Linear Osmosis filter). A *(linear) osmosis filter* is a drift-diffusion equation with initial condition f and homogeneous Neumann conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \operatorname{div}(\mathbf{d}u), & \text{on } \Omega \times (0, T] \\ u(\mathbf{x}, 0) = f(\mathbf{x}), & \text{on } \Omega \\ \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0, T] \end{cases} \tag{5.1}$$

where $\mathbf{d} : \Omega \to \mathbb{R}^2$ is the *drift vector field*.

This equation provides some nice properties:

- **Preservation of the Average Grey Value**: this is essential in segmentation algorithms and in medical applications, because grey values measure often physical qualities of the depicted object:

$$\frac{1}{|\Omega|} \int_\Omega u(\mathbf{x}, t)\, \mathrm{d}\mathbf{x} = \frac{1}{|\Omega|} \int_\Omega f(\mathbf{x})\, \mathrm{d}\mathbf{x} \quad \forall t > 0.$$

Proof. We denote the average gray value at time $t \geq 0$ with

$$\mu(t) = \frac{1}{|\Omega|} \int_\Omega u(\mathbf{x}, t)\, \mathrm{d}\mathbf{x}.$$

Using the divergence Theorem and the homogeneous Neumann conditions we write

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{1}{|\Omega|} \int_\Omega \partial_t u(\mathbf{x}, t)\, \mathrm{d}\mathbf{x} = \frac{1}{|\Omega|} \int_\Omega \operatorname{div}(\nabla u - \mathbf{d}u)\, \mathrm{d}\mathbf{x} = \frac{1}{|\Omega|} \int_{\partial\Omega} \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle\, \mathrm{d}S = 0.$$

So the average gray value remains constant over time. □

- **Preservation of Positivity**: starting from a positive image, we have guaranteed a weaker property than the minimum-maximum principle. Otherwise the *(Linear) Osmosis filter* can violate this property. In other words, if $u(\mathbf{x},0)>0$, then $u(\mathbf{x},t)>0$, for all $\mathbf{x}\in\Omega$, $\forall t>0$.

- **Convergence to a Nontrivial Steady State**: if $\mathbf{d}$ satisfies

$$\mathbf{d}=\nabla(\ln v)=\frac{\nabla v}{v},$$

with some positive image v, osmosis process converges to v up to a multiplicative constant which ensures preservation of the average grey value of u. Thus, osmosis creates nontrivial steady states. This is a fundamental difference to diffusion that allows only flat steady states. So the steady state equation $\Delta u-\operatorname{div}(\mathbf{d}u)=0$ is equivalent to the Euler-Lagrange Equation of the energy functional

$$E(u)=\int_\Omega v\left|\nabla\left(\frac{u}{v}\right)\right|^2.$$

Proof. This follows from a simple computation in Calculus of Variations:

$$\begin{aligned}
0&=\frac{\partial}{\partial t}\bigg|_{t=0}\left(\int_\Omega v\left|\nabla\left(\frac{u+t\varphi}{v}\right)\right|^2\mathrm{d}\mathbf{x}\right)=\frac{\partial}{\partial t}\bigg|_{t=0}\left(\int_\Omega v\left|\nabla\left(\frac{u}{v}\right)+t\,\nabla\left(\frac{\varphi}{v}\right)\right|^2\mathrm{d}\mathbf{x}\right)\\
&=\frac{\partial}{\partial t}\bigg|_{t=0}\left(\int_\Omega v\left|\nabla\left(\frac{u}{v}\right)\right|^2\mathrm{d}\mathbf{x}+2t\int_\Omega v\,\nabla\left(\frac{u}{v}\right)\cdot\nabla\left(\frac{\varphi}{v}\right)\mathrm{d}\mathbf{x}+t^2\int_\Omega\left|\nabla\left(\frac{\varphi}{v}\right)\right|^2\mathrm{d}\mathbf{x}\right)\\
&=2\int_\Omega v\,\nabla\left(\frac{u}{v}\right)\cdot\nabla\left(\frac{\varphi}{v}\right)\mathrm{d}\mathbf{x}=2\left\langle v\,\nabla\left(\frac{u}{v}\right),\nabla\left(\frac{\varphi}{v}\right)\right\rangle_\Omega
\end{aligned}$$

then, passing to the adjoint it gives

$$=-2\left\langle\operatorname{div}\left(v\,\nabla\left(\frac{u}{v}\right)\right),\frac{\varphi}{v}\right\rangle_\Omega=0\implies-\frac{2}{v}\operatorname{div}\left(v\,\nabla\left(\frac{u}{v}\right)\right)=0.$$

So, this converges to the steady state equation $\Delta u-\operatorname{div}(\mathbf{d}u)=0$:

$$\begin{aligned}
0&=\operatorname{div}\left(v\,\nabla\left(\frac{u}{v}\right)\right)=\operatorname{div}\left(v\frac{\nabla(u)v-u\,\nabla(v)}{v^2}\right)=\operatorname{div}\left(\frac{\nabla(u)v-u\,\nabla(v)}{v}\right)\\
&=\operatorname{div}\left(\nabla(u)-\frac{u\,\nabla(v)}{v}\right)=\Delta u-\operatorname{div}(\mathbf{d}u).
\end{aligned}$$

□

Remark 5.3. We noted that the steady state equation $\Delta u-\operatorname{div}(\mathbf{d}u)=0$ is the same Equation (4.12) to be solved for *Non-local Poisson approach* in inpainting problem. In this case, $\mathbf{d}$ is the *guidance vector field*. Since $\mathbf{d}$ contains the gradient information of $\ln v$, osmosis is a process for data integration.

Applying the osmosis theory to each RGB color channel, we can simply extend these results from grayscale to color images, computing the drift vectors for each channel.

5.2 The discrete model

Given a 2D image u, we can discretize Equation (5.1) using a grid with a space step $h = 1$ in each direction $\hat{\imath}$ and $\hat{\jmath}$ and a small enough time step dt. We denote with $u^t_{i,j}$ a pixel-color approximation to u in the grid $\left(\left(i-\frac{1}{2}\right)h, \left(j-\frac{1}{2}\right)h\right)$ at time t. Setting $\mathbf{d} = (d_1, d_2)^{\mathrm{T}}$, the first attempt to solve Equation (5.1) is based on a 2D straightforward finite difference discretization. So,

$$\frac{\partial u}{\partial t} = \Delta u - \operatorname{div}(\mathbf{d}u) = \Delta u - \operatorname{div}\mathbf{d}u - \mathbf{d}\cdot\nabla u = \Delta u - \frac{\partial d_1}{\partial x}u - \frac{\partial d_2}{\partial y}u - d_1\frac{\partial u}{\partial x} - d_2\frac{\partial u}{\partial y}$$

is discretized as

$$\begin{aligned}
\frac{\partial u_{i,j}}{\partial t} &= \frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2} - \frac{d_{1,i+\frac{1}{2},j}-d_{1,i-\frac{1}{2},j}}{h}u_{i,j} - \frac{d_{2,i,j+\frac{1}{2}}-d_{2,i,j-\frac{1}{2}}}{h}u_{i,j} \\
&\quad - d_1\frac{u_{i+1,j}-u_{i,j}+u_{i,j}-u_{i-1,j}}{2h} - d_2\frac{u_{i,j+1}-u_{i,j}+u_{i,j}-u_{i,j-1}}{2h} \\
&= \frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2} - \frac{d_{1,i+\frac{1}{2},j}-d_{1,i-\frac{1}{2},j}}{h}u_{i,j} - \frac{d_{2,i,j+\frac{1}{2}}-d_{2,i,j-\frac{1}{2}}}{h}u_{i,j} \\
&\quad - d_1\frac{u_{i+1,j}-u_{i,j}}{2h} - d_1\frac{u_{i,j}-u_{i-1,j}}{2h} - d_2\frac{u_{i,j+1}-u_{i,j}}{2h} - d_2\frac{u_{i,j}-u_{i,j-1}}{2h} \\
&= \frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2} - \frac{d_{1,i+\frac{1}{2},j}-d_{1,i-\frac{1}{2},j}}{h}u_{i,j} - \frac{d_{2,i,j+\frac{1}{2}}-d_{2,i,j-\frac{1}{2}}}{h}u_{i,j} \\
&\quad - d_{1,i+\frac{1}{2},j}\frac{u_{i+1,j}-u_{i,j}}{2h} - d_{1,i-\frac{1}{2},j}\frac{u_{i,j}-u_{i-1,j}}{2h} \\
&\quad - d_{2,i,j+\frac{1}{2}}\frac{u_{i,j+1}-u_{i,j}}{2h} - d_{2,i,j-\frac{1}{2}}\frac{u_{i,j}-u_{i,j-1}}{2h} \\
&= \frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2} \\
&\quad - d_{1,i+\frac{1}{2},j}\frac{2u_{i,j}+u_{i+1,j}-u_{i,j}}{2h} - d_{1,i-\frac{1}{2},j}\frac{-2u_{i,j}+u_{i,j}-u_{i-1,j}}{2h} \\
&\quad - d_{2,i,j+\frac{1}{2}}\frac{2u_{i,j}+u_{i,j+1}-u_{i,j}}{2h} - d_{2,i,j-\frac{1}{2}}\frac{-2u_{i,j}+u_{i,j}-u_{i,j-1}}{2h} \\
&= \frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2} \\
&\quad - d_{1,i+\frac{1}{2},j}\frac{u_{i,j}+u_{i+1,j}}{2h} - d_{1,i-\frac{1}{2},j}\frac{-u_{i,j}-u_{i-1,j}}{2h} \\
&\quad - d_{2,i,j+\frac{1}{2}}\frac{u_{i,j}+u_{i,j+1}}{2h} - d_{2,i,j-\frac{1}{2}}\frac{-u_{i,j}-u_{i,j-1}}{2h} \\
&= \frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2} \\
&\quad - \frac{1}{h}\left(d_{1,i+\frac{1}{2},j}\frac{u_{i,j}+u_{i+1,j}}{2} - d_{1,i-\frac{1}{2},j}\frac{u_{i,j}+u_{i-1,j}}{2}\right) \\
&\quad - \frac{1}{h}\left(d_{2,i,j+\frac{1}{2}}\frac{u_{i,j}+u_{i,j+1}}{2h} - d_{2,i,j-\frac{1}{2}}\frac{u_{i,j}+u_{i,j-1}}{2h}\right),
\end{aligned}$$

or, written as penta-diagonal discretization,

$$\frac{\partial u_{i,j}}{\partial t} = \left(\frac{1}{h^2} - \frac{d_{1,i+\frac{1}{2},j}}{2h}\right) u_{i+1,j} + \left(\frac{1}{h^2} + \frac{d_{1,i-\frac{1}{2},j}}{2h}\right) u_{i-1,j}$$
$$+ \left(\frac{1}{h^2} - \frac{d_{2,i,j+\frac{1}{2}}}{2h}\right) u_{i,j+1} + \left(\frac{1}{h^2} + \frac{d_{2,i,j-\frac{1}{2}}}{2h}\right) u_{i,j-1}$$
$$+ \left(-\frac{4}{h^2} - \frac{d_{1,i+\frac{1}{2},j}}{2h} + \frac{d_{1,i-\frac{1}{2},j}}{2h} - \frac{d_{2,i,j+\frac{1}{2}}}{2h} + \frac{d_{2,i,j-\frac{1}{2}}}{2h}\right) u_{i,j}. \tag{5.2}$$

where, for some positive image v, the discrete approximation of the drift-vector $\mathbf{d} = (d_1, d_2)^{\mathrm{T}} = \frac{\nabla v}{v}$ at intermediate grid point is:

$$d_{1,i+\frac{1}{2},j} = \frac{2(v_{i+1,j} - v_{i,j})}{h(v_{i+1,j} + v_{i,j})}, \quad \text{and} \quad d_{2,i,j+\frac{1}{2}} = \frac{2(v_{i,j+1} - v_{i,j})}{h(v_{i,j+1} + v_{i,j})}.$$

Remark 5.4. Guessing to approximate a discontinued image u with finite difference method, can not assure the right spatial convergence rate because we should assume $u \in C^4$ for a second order operator.

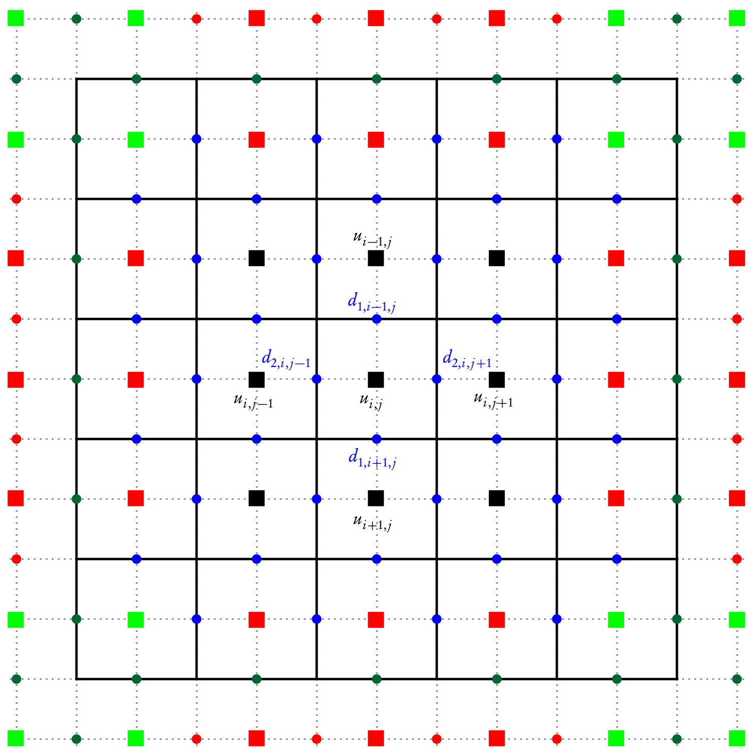


Figure 15: The black grid is a simple representation of a 5×5 image. Each ■ is a pixel in internal domain of the image, each ■ is a mirrored boundary point and each ■ is a special mirrored boundary point at the corners. Each • is a zero drift vector point (automatically imposed) while each other • and • is automatically computed.

To ensure that in Equation (5.2) the weights of all four neighbours of $u_{i,j}$ are positive, we restrict to drift vector fields $(d_1(\mathbf{x}), d_2(\mathbf{x}))^{\mathrm{T}}$, with

$$|d_1(\mathbf{x})| < \frac{2}{h} \quad \text{and} \quad |d_2(\mathbf{x})| < \frac{2}{h}, \quad \forall \mathbf{x} \in \Omega.$$

With this scheme, we can compute the solution at timestep $t+1$ given the solution at timestep t. This scheme also holds for boundary points mirroring the image at its boundaries and assume a zero drift vector across boundaries. Figure 15 shows very well the situation for a simple 5×5 image.

5.3 The θ-method

The θ-method is a very popular method in Numerical Analysis to solve a PDE at a desired timestep T based on updating the solution at time t with a desired timestep dt. We discretize the linear problem

$$\partial_t u = \Delta u - \operatorname{div}(\mathbf{d}u) \quad \text{as} \quad \frac{u^{t+1} - u^t}{\mathrm{dt}} = P(\theta u^{t+1} + (1-\theta)u^t), \tag{5.3}$$

where $P(\,\cdot\,)$ is the pentadiagonal matrix discretization of $\Delta(\,\cdot\,) - \operatorname{div}(\mathbf{d}\,\cdot\,)$. Starting from u^0, we can obtain the solution at the desired time T updating

$$u^{t+1} = (I - \mathrm{dt}\theta P)^{-1}(I + \mathrm{dt}(1-\theta)P)u^t \tag{5.4}$$

until T (or a steady state condition) is reached. Different choices of $\theta \in [0,1]$ are allowed:

- Forward Euler ($\theta = 0$): an explicit method which converges only if the timestep satisfies

$$\mathrm{dt} < \frac{h^2}{8}.$$

  This is too expensive for solving Equation (5.4), because the choices allowed for dt are too small.

- Crank - Nicholson ($\theta = 0.5$): an implicit method (or *trapezoid rule*). The expected order is 2.

- Backward Euler ($\theta = 1$): an implicit method. The expected order is 1.

Once θ is chosen, we have to update u^t: this can be performed with direct or iterative methods but the system (5.1) is non-symmetric so we can't take advantages from very popular tools in numerical analysis.

Direct method: the LUpq factorization

The time updating equation (5.4) can be solved with a variant of **LU** decomposition, called **LUpq** which takes advantage from permutations on rows without specifying directly the permutation matrix: this typically requires less time and less storage space respect the classical L,U,P factorization:

```
[L,U,p,q]=lu(I-dt*theta*A,'vector');
B = (I+dt*(1-theta)*A);
for t = (dt:dt:T)
    C = B*y;
    y(q) = U\(L\(noto(p)));
end
```

Iterative method: BiCGStab solver

Because of the dimension of the problem, an iterative method can be used to solve the linear system (5.4). According to the experiments in [Vogel *et al.*(2013)], the iterative method BiCGStab, *Biconjugate gradient stabilized method* seems to be the faster way to solve the linear system (5.2) with θ-method (5.4). The key core of iteratives methods, is starting from an approximate solution modifying, at every iteration, one or more components and trying to minimize the residual norm. For example, if u^t is the solution at step t, we want to minimize $Au^t - b$. More details about the BiCGStab methods used in our experiments can be found in Section 5.6.

5.4 A Fourier approach

Obviously, the most difficult task in solving Equation (5.4) is to invert the matrix $(I - \mathrm{dt}\theta P)$. This could be difficult in the first instance because of unsymmetry of the problem, but we can take advantage of a Fourier method to invert the term containing the laplacian information. The main drawback of this method is that it works well for periodic functions or data: in our context image is not periodic and this could introduce wrong Fourier coefficients. For this reason we mirror the image u as in Figure 16 to have a periodic signal.

Considering $P = \Delta + D$, with Δ as laplacian and D as $-\mathrm{div}(\mathbf{d}u)$, we can split θ in Equation (5.3) in θ_1 and θ_2, resulting:

$$(I - \mathrm{dt}\theta_1 \Delta - \mathrm{dt}\theta_2 D)u^{t+1} = (I + \mathrm{dt}(1-\theta_1)\Delta + \mathrm{dt}(1-\theta_2)D)u^t.$$

Finally, setting $\theta_1 = 1$ and $\theta_2 = 0$, we obtain $(I - \mathrm{dt}\theta_1 \Delta)u^{t+1} = (I + \mathrm{dt}D)u^t$, so

$$u^{t+1} = (I - \mathrm{dt}\,\Delta)^{-1}(I + \mathrm{dt}D)u^t. \tag{5.5}$$

Through a Fourier collocation method we can solve immediately Equation (5.5) writing the solution u and Δu in the truncated Fourier series

$$u = \sum_{|k|_\infty \le N} u_k \mathrm{e}^{2\mathrm{i}\pi x\cdot k} \quad \text{and} \quad \Delta u = -\sum_{|k|_\infty \le N} u_k 4\pi^2 |k|^2 \mathrm{e}^{2i\pi x\cdot k},$$

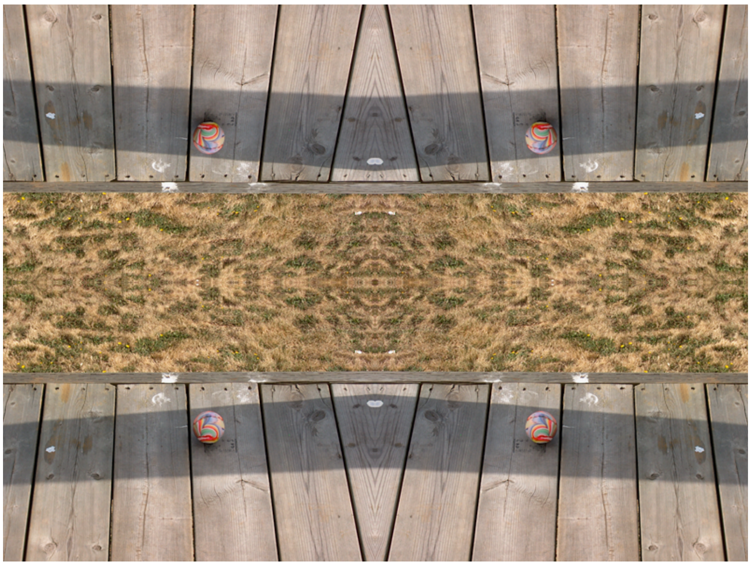


Figure 16: Image mirrored to have the correct periodic boundary conditions.

and introducing v such that:

$$\sum_{|k|_\infty \le N} v_k e^{2i\pi x\cdot k} = v = (I - \mathrm{dt}\,\Delta)u = \sum_{|k|_\infty \le N} (1 + \mathrm{dt}4\pi^2|k|^2)u_k e^{2i\pi x\cdot k}.$$

So, for every k, we obtain $v_k = (1 + \mathrm{dt}4\pi^2|k|^2)u_k$ from which:

$$u_k = \frac{v_k}{(1 + \mathrm{dt}4\pi^2|k|^2)},$$

and the update step is trivial (Algorithm 3). We also note that laplacian coefficients are calculated once: there is no need to iterate this computation. Visually speaking, evaluating D with finite differences, produce an acceptable solution despite of a perceptible error from the reference solution (Figure 18e).

Algorithm 3 Semi-implicit solver with bridge Fourier collocation

Input: u^0 (original mirrored image, 2D matrix of N rows and M columns), k pixel-indexes.

Output: u^T at time $T = t^{\text{end}}$.

1: Evaluate the drift term D with finite difference;
2: $\text{coeff} = (1 + 4k\pi^2\mathrm{dt})$;
3: $A = (I + \mathrm{dt}D)$;
4: **for** $t = \mathrm{dt} : \mathrm{dt} : T$ **do**
5: $\quad u = \text{reshape}(Au(:), N, M)$;
6: $\quad \hat{v} = \text{fft2}(u)./\text{coeff}$;
7: $\quad u = \text{ifft2}(\hat{v})$;
8: **end for**

Another Fourier-based implementation is shown in Algorithm 4: since a differentiation in the space domain is a multiplication with $(ik)^d$ in frequency domain (d is the order of the differentiation), we can write a semi-implicit scheme, with an explicit calculus of div($\mathbf{d}u$):

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - \operatorname{div}(\mathbf{d}u) \\
\frac{u^{t+1}-u^{t}}{\mathrm{dt}} &= \Delta u^{t+1} - \operatorname{div}(\mathbf{d}u^{t}) \\
u^{t+1}-u^{t} &= \mathrm{dt}\,\Delta u^{t+1} - \mathrm{dt}\operatorname{div}(\mathbf{d}u^{t}) \\
u^{t+1}-\mathrm{dt}\,\Delta u^{t+1} &= u^{t} - \mathrm{dt}\operatorname{div}(\mathbf{d}u^{t}).
\end{aligned}$$

Algorithm 4 Semi-implicit solver with fully Fourier collocation

Input: u^0 (original image, 2D matrix of N rows and M columns), k pixel-indexes.

Output: u^T at time $T = t^{\mathrm{end}}$.

define the `flag_log` $= \{0, 1\}$ variable, useful to change the computation of $\mathbf{d}$.
if `flag_log` **then**
    $\mathbf{d} = \nabla \log u = \mathrm{ifft2}(\mathrm{fft2}(\log u). * (2\pi i k))$;
else
    $\nabla u = \mathrm{ifft2}(\mathrm{fft2}(u). * (2\pi i k))$;
    $\mathbf{d} = \nabla u./u$;
end if
eventually $\mathbf{d} = \mathbf{d}. * u_{\mathrm{mask}}$ in shadow application;
$\mathrm{coeff} = (1 - 4k\pi^2\mathrm{dt})$;
for $t = \mathrm{dt} : \mathrm{dt} : T$ **do**
    $\operatorname{div}(\mathbf{d}u) = \mathrm{ifft2}((\mathrm{fft2}(\mathbf{d}u). * (2\pi i k)))$;
    $u = \mathrm{ifft2}(\mathrm{fft2}(u - \mathrm{dt}\operatorname{div}(\mathbf{d}u))./\mathrm{coeff})$;
end for

The major advantages of these Fourier semi-implicit methods are the speed up in computational time while the errors from what we choose as reference solution are concentrated only on big jumps of color: the upper-bound error allows bigger timesteps than BiCGSstab when solving. This arises from our experimental results (see Section 5.6).

5.5 Exponential Integrators

The osmosis filter in Equation (5.1) can be also solved as a general differential equation of this type:

$$\begin{cases} y'(t) = ay(t) + b, & t > t_0 \\ y(t_0) = y_0, \end{cases}$$

where the analytic solution is given from the *variation of constants* method:

$$y(t)=\mathrm{e}^{(t-t_0)a}y_0+\int_{t_0}^{t}\mathrm{e}^{(t-\tau)a}b(\tau,y(\tau))\,\mathrm{d}\tau.$$

In fact

$$\begin{aligned}y'(t)&=a\mathrm{e}^{(t-t_0)a}y_0+a\int_{t_0}^{t}\mathrm{e}^{(t-\tau)a}b(\tau,y(\tau))\,\mathrm{d}\tau+\mathrm{e}^{(t-t)a}b(t,y(t))\\&=ay(t)+b(t,y(t)).\end{aligned}$$

Remark 5.5. We observe that

$$\begin{aligned}\int_{t_0}^{t}\mathrm{e}^{(t-\tau)a}\,\mathrm{d}\tau&=-\frac{1}{a}\int_{t_0}^{t}-a\mathrm{e}^{(t-\tau)a}\,\mathrm{d}\tau=-\frac{1}{a}\mathrm{e}^{(t-\tau)a}\Big|_{t_0}^{t}=-\frac{1}{a}\left(1-\mathrm{e}^{(t-t_0)a}\right)\\&=(t-t_0)\frac{\mathrm{e}^{(t-t_0)a}-1}{(t-t_0)a}=(t-t_0)\varphi_1((t-t_0)a)\quad\text{where}\quad\varphi_1(z)=\frac{\mathrm{e}^z-1}{z}=\sum_{j=0}^{\infty}\frac{z^j}{(j+1)!}.\end{aligned}$$

Remark 5.6. Let $A\in\mathbb{R}^{n\times n}$ a square matrix. We consider

$$\exp(A)=\sum_{j=0}^{\infty}\frac{A^j}{j!}.$$

There are many different ways to compute $\exp(A)$. In our case, A is a big sparse matrix. Moreover we don't want an explicit result for $\exp(A)$ but we need $\exp(A)$ to perform the matrix-vector multiplication $\exp(A)v$, given v. The well know *Krylov* methods avoid to compute explicitly $\exp(A)$ providing instead $\exp(A)v$. Based on the Arnoldi technique, we can decompose A in

$$A=V_mH_mV_m^T\implies V_m^TAV_m=H_m,$$

with $V_m\in\mathbb{R}^{n\times m}$ and $V_m^TV_m=I$, $V_me_1=v$ and H_m the *almost* triangular Hessenberg matrix of order m (with $m\ll n$). Then $AV_m\approx V_mH_m$ so

$$\exp(A)V_m\approx V_m\exp(H_m)\implies\exp(A)v\approx V_m\exp(H_m)e_1$$

where $\exp(H_m)$ is computed via Padè approximation. More details in [Caliari(2013)].

Given the differential equation

$$\begin{cases}\mathbf{y}'(t)=A\mathbf{y}(t)+\mathbf{b}(t,\mathbf{y}(t)), & t>t_0\\\mathbf{y}(t_0)=\mathbf{y}_0,\end{cases}\tag{5.6}$$

the analytic solution can be written as

$$\mathbf{y}(t)=\exp((t-t_0)A)\mathbf{y}_0+\int_{t_0}^{t}\exp((t-\tau)A)\mathbf{b}(\tau,\mathbf{y}(\tau))\,\mathrm{d}\tau,$$

or, written in another way,

$$\mathbf{y}(t)=\exp((t-t_0)A)\mathbf{y}_0+(t-t_0)\varphi_1((t-t_0)A)\mathbf{b}=\mathbf{y}_0+(t-t_0)\varphi_1((t-t_0)A)(A\mathbf{y}_0+\mathbf{b}).$$

Proof. It is trivial to check $\mathbf{y}(t_0) = y_0$. Moreover, from

$$\frac{\mathrm{d}[(t-t_0)\varphi_1((t-t_0)A)\mathbf{b}]}{\mathrm{d}t} = \exp((t-t_0)A)\mathbf{b} = (t-t_0)A\varphi_1((t-t_0)A)\mathbf{b} + \mathbf{b},$$

we have

$$\begin{aligned}\mathbf{y}'(t) &= A\exp((t-t_0)A)\mathbf{y}_0 + \exp((t-t_0)A)\mathbf{b}\\ &= A[\exp((t-t_0)A)\mathbf{y}_0 + (t-t_0)\varphi_1((t-t_0)A)\mathbf{b}] + \mathbf{b} = A\mathbf{y}(t) + \mathbf{b}.\end{aligned}$$

□

So, we can solve the differential Equation (5.6) with the *Euler exponential method*:

$$\mathbf{y}_{n+1} = \exp(\mathrm{d}tA)\mathbf{y}_n + \mathrm{d}t\varphi_1(\mathrm{d}tA)b(t_n,\mathbf{y}_n) = \mathbf{y}_n + \mathrm{d}t\varphi_1(\mathrm{d}tA)(A\mathbf{y}_n + \mathbf{b}(t_n,\mathbf{y}_n)).$$

Proposition 5.7. *The Euler exponential method is* exact *if $b(\mathbf{y}(t)) = b(\mathbf{y}_0) \equiv b$ or of order one otherwise.*

Proof. We have

$$\mathbf{y}_{n+1} = \exp(\mathrm{d}tA)\mathbf{y}_n + \int_{t_n}^{t_{n+1}} \exp((t_{n+1}-\tau)A)\mathbf{b}(t_n,\mathbf{y}_n)\,\mathrm{d}\tau$$

Let $\mathbf{g}(t) = b(t,\mathbf{y}(t))$, using the variation of constants method we have

$$\begin{aligned}\mathbf{y}(t_{n+1}) &- \exp(\mathrm{d}tA)\mathbf{y}(t_n) - \int_{t_n}^{t_{n+1}} \exp((t_{n+1}-\tau)A)\mathbf{g}(t_n)\,\mathrm{d}\tau\\ &= \exp(\mathrm{d}tA)\mathbf{y}(t_n) + \int_{t_n}^{t_{n+1}} \exp((t_{n+1}-\tau)A)\mathbf{g}(\tau)\,\mathrm{d}\tau\\ &\quad - \exp(\mathrm{d}tA)\mathbf{y}(t_n) - \int_{t_n}^{t_{n+1}} \exp((t_{n+1}-\tau)A)\mathbf{g}(t_n)\,\mathrm{d}\tau\\ &= \int_{t_n}^{t_{n+1}} \exp((t_{n+1}-\tau)A)(\mathbf{g}(t_n) + \mathbf{g}'(\tau_n)(\tau - t_n) - \mathbf{g}(t_n))\,\mathrm{d}\tau\\ &= \mathrm{d}t^2\varphi_2(\mathrm{d}tA)\mathbf{g}'(\tau_n) = \mathcal{O}(\mathrm{d}t^2), \quad \text{with } \varphi_2(z) = \frac{\mathrm{e}^z - 1 - z}{z^2} = \sum_{j=0}^{\infty}\frac{z^j}{(j+2)!}.\end{aligned}$$

□

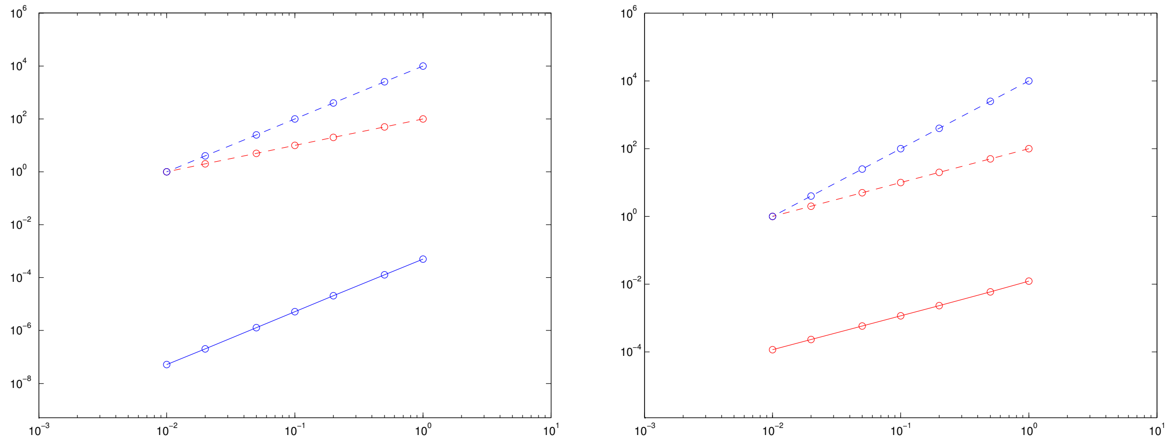
For our tests we use the Matlab package in [Al-Mohy and Higham(2011)] to exponentiate the matrix and we used this solution as reference solution in Tables 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6: the function used is `expmv.m`, which is the dedicate function to perform the matrix-vector multiplication with Exponential Integrators method, whose precision (exact in time from theory) depends only from the machine precision. We compare also this result with another implementation of Exponential Integrators found in [Sidje(1998)]: the script used here is `expv.m`.

5.6 Numerical results

Experiments of Tables 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6 are performed on Mac OSX 10.8.5, 2.2 GHz Intel Core i7 Quad-Core, RAM 4GB 1333 MHz DDR3, with Matlab 2012a 64 bit.

Order of θ-method

An important question arising when solving a partial different equation is testing the code in order to produce the true results with the expected accuracy. In this case we don't know the exact solution: as far as we know, evolving Equation (5.1) in time converges to a steady state. For this reason we computed a *reference solution* with the exponential method, which is *exact* in time. Another solution could be evaluate the solution at a finer timestep and estimate the errors on a much coarser timestep. In this case, the timestep $\mathrm{dt}_{\mathrm{ref}}$ associated to the *reference solution* must satisfy $\mathrm{dt}_{\mathrm{ref}} \ll \mathrm{dt}$, for any timestep dt used to compute the other solutions.



(a) LUpq, $\theta = 0.5$: we obtain a 2-order slope (blue). (b) LUpq, $\theta = 1$: we obtain a 1-order slope (red).

Figure 17: Log-log plot of θ-method's time order for shadow removal application presented in the next pages: 1-order slope in red-dashed line while 2-order slope in blu-dashed line.

Remark 5.8. Estimating the error and testing the order of accuracy by this approach only confirm that the code is converging very nicely to *some* function with the desired rate. It is very possible that the code is converging to the *wrong* function (details in [LeVeque(2007)]). This is not the case so we leave to the evaluation of how good is the *function-image* solution to the human visual system (our eyes).

Error of Fourier methods

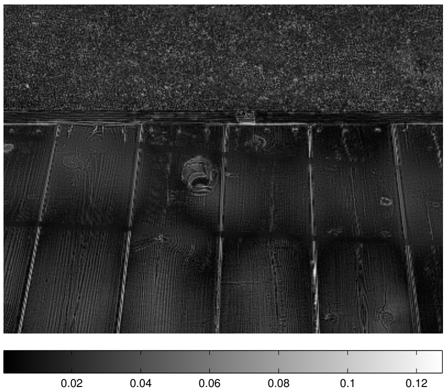
In Algorithm 4 we reported two different ways to evaluate the drift vector field $\mathbf{d}$. From our experiments, we noted that the second choice of $\mathbf{d}$, i.e. $\mathbf{d} = \nabla u / u$, is more stable in the sense that the difference from the reference solution computed with the Exponential Method is concentrated only on shadow boundaries. This clearly arise from visual consideration on Figure 18. For this reasons in Tables 5.1, 5.2, 5.3, 5.4, 5.5, 5.6 and in Figure 21f we consider only the drift vector field $\mathbf{d}$ evaluated with $\mathbf{d} = \nabla u / u$ when using Algorithm 4. Changing of dt produce only a different *error* from reference

(a) Reference with `expmv`.

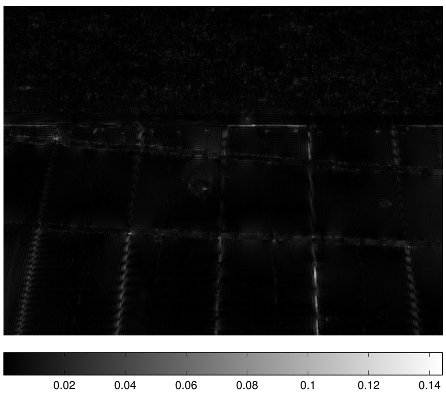
(b) Alg. 3.

(c) Alg. 4 with $\mathbf{d} = \nabla \log u$.

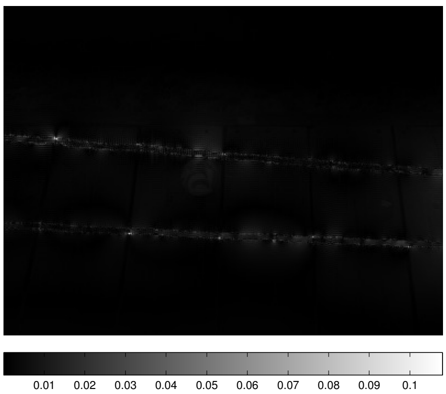
(d) Alg. 4 with $\mathbf{d} = \nabla u / u$.



(e) Error Alg. 3.



(f) Error Alg. 4 with $\mathbf{d} = \nabla \log u$.



(g) Error Alg. 4 with $\mathbf{d} = \nabla u / u$.

Figure 18: Fourier results for Shadow Removal (Section 5.7.1) at $T = 1000$ with timestep $\mathrm{dt} = 100$.

solution outside shadow boundary: the maximum error doesn't seems to change due to the different operator acting on shadow boundaries (only laplacian) from the rest of the image (laplacian with divergence): this is in somehow relation to the Gibbs phenomenon.

Remark 5.9. It seems quite obvious that $\mathbf{d} = \nabla \log u$ produce a slightly worse error difference respect to $\mathbf{d} = \nabla u / u$: in fact, the reference solution, evaluated with Exponential Method, is computed by the finite difference matrix in the space domain, which is based on the discretization of $\mathbf{d}$ as $\nabla u/u$ instead of computing directly the discretization of $\mathbf{d} = \nabla \log u$. For this reason we can't predict which method is the best one although we expect that computing $\mathbf{d} = \nabla \log u$ is more coherent with the original formulation of the system (5.1).

Even if Algorithm 3 seems to produce the worst (but not too distant from the expected solution) result in Figure 18e, it is the fastest method tested overall (see Tables 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6): for this reason we claim that for a commercial use of this filter in software products, this is the best choice because the result 18b is very similar to what we expect. We motivated this error difference from the reference solution with the separation of the *transport* operator from the *diffusion* one.

BiCGStab and variants

In solving Equation (5.4) with BiCGStab, we used various tolerances, summarized in Tables 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6. The maximum number of iteration allowed are maxit $= 30$ and the initial timestep is dt $= 1$. Different from [Vogel *et al.*(2013)], we rescaled our data in $[\varepsilon, 255+\varepsilon]/(255+\varepsilon)$, with $\varepsilon = 1$, insted of $[\varepsilon, 255+\varepsilon]$ and we assume that the Ground Truth is unknown: for this reason we need a different stopping criterion respect to to that used in [Vogel *et al.*(2013)]. In particular, this is the most common case where the Ground Truth depends only by the visual expectations of the user.

We compare BiCGStab in four variants:

- BiCGStab: the classical BiCGStab method, with fixed timestep dt with updating solution until a fixed time T is reached;

- A-BiCGStab: as BiCGStab but we used the starting dt only for the first computation; then timestep can be update at every computation:

  1. we compute an average iteration value desired for BiCGStab with `averit = 35*maxit/50`, and a safe zone where dt will not be changed:

$$\texttt{safe\_zone} = [0.8 \cdot \texttt{averit}, 1.2 \cdot \texttt{averit}];$$

2. then we update dt until T is reached with the following experimental rule:

$$\mathrm{dt}(t+1) = \begin{cases} 1.2\cdot\mathrm{dt}(t) & \text{if BiCGStab converges in } k < \min(\texttt{safe\_zone}) \text{ steps;} \\ 1.0\cdot\mathrm{dt}(t) & \text{if BiCGStab converges in } k \in \texttt{safe\_zone} \text{ steps;} \\ 0.8\cdot\mathrm{dt}(t) & \text{if BiCGStab converges in } k > \max(\texttt{safe\_zone}) \text{ steps;} \\ 0.5\cdot\mathrm{dt}(t) & \text{otherwise (don't increasing } t\text{);} \end{cases} \tag{5.7}$$

- F-BiCGStab: as BiCGStab but the alghoritm stops only when the difference between the new solution and the previous one is under a fixed tolerance (weighted by the fixed timestep):

```
norm(y_new-y)/norm(y_new) < dt * tol_exit;
```

- FA-BiCGStab: as A-BiCGStab but the algorithm stops only when the difference between the new solution and the previous one is under a fixed tolerance (weighted by the *current* timestep):

```
norm(y_new-y)/norm(y_new) < dt(t) * tol_exit;
```

Because of BiCGStab can be called in Matlab with several different parameters, we tested some different situations: firstly, we focused our attention on the difference between the following command lines

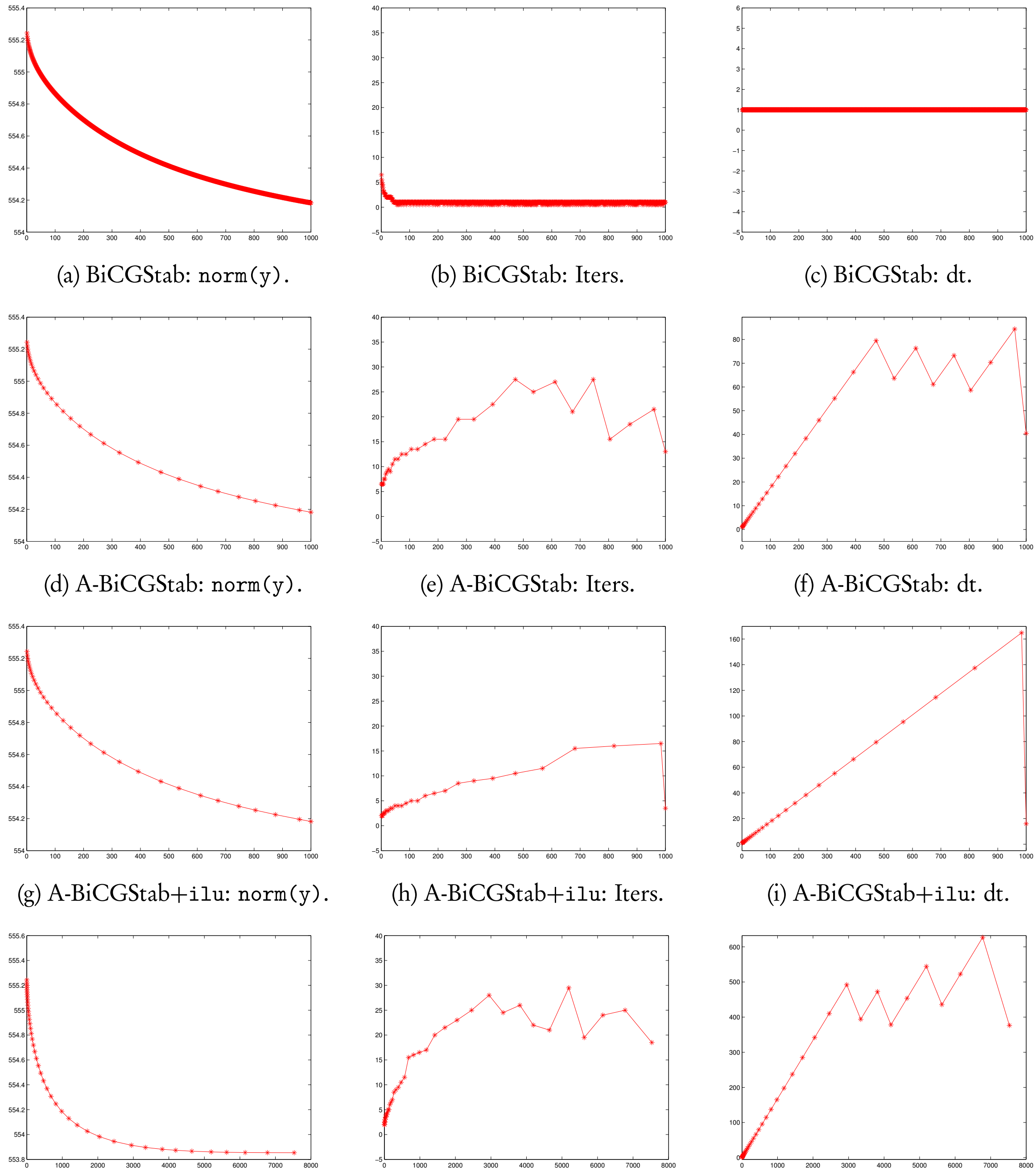
```
y = bicgstab(I-dt*theta*A,(I + dt*(1-theta)*A)*y,tol,maxit);
y = bicgstab(I-dt*theta*A,(I + dt*(1-theta)*A)*y,tol,maxit,[],[],y);
```

where the second line refers to BiCGStab with an initial guessing vector (or solution at the previous timestep): during our tests, the second command line performed better than the first one, where BiCGStab starts with a zero guessing vector: this is reasonable because the initial guess vector (or solution at the previous timestep) is *near*, in some sense, to the expected solution and this shorts iterations and computation time. For this reason we skipped to report the results from the first command line in Tables 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6.

Then we use a preconditioner in order to speed up the computations: unfortunately, the matrix is unsymmetric so we are forced to use the `ilu` function which performs a sparse incomplete LU factorization (a unit lower triangular matrix and an upper triangular matrix): for this reason, Tables 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6 reports the outputs from the following command lines:

```
y = bicgstab(I-dt*theta*A,(I + dt*(1-theta)*A)*y,tol,maxit,[],[],y);
y = bicgstab(I-dt*theta*A,(I + dt*(1-theta)*A)*y,tol,maxit,L,U,y);
```

Visually speaking, when comparing the solution obtained with F-BiCGStab and FA-BiCGStab, we observed a more reliable result with the preconditioning matrix because the greedy algorithm



(a) BiCGStab: `norm(y)`. (b) BiCGStab: Iters. (c) BiCGStab: dt.

(d) A-BiCGStab: `norm(y)`. (e) A-BiCGStab: Iters. (f) A-BiCGStab: dt.

(g) A-BiCGStab+`ilu`: `norm(y)`. (h) A-BiCGStab+`ilu`: Iters. (i) A-BiCGStab+`ilu`: dt.

(j) FA-BiCGStab+`ilu`: `norm(y)`. (k) FA-BiCGStab+`ilu`: Iters. (l) FA-BiCGStab+`ilu`: dt.

Figure 19: Different calls of BiCGStab to Shadow Removal problem to prove the adaptivity of the algorithm. Results form Table 5.3 with $\theta = 0.5$ (only red channel). We notice the convergence to a steady state and the adaptivity of the timestep as imposed in Equation (5.7). In particular, we compare Figures 19e, 19h and 19k: these clearly show the oscillation on the average valued imposed. Figures 19i, 19f and 19l report the length of timestep used: this confirm the adaptivity of the proposed methods.

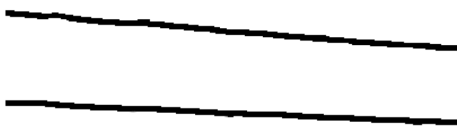
written under the rule of Equation (5.7) seems to use bigger timesteps when preconditioning (see Figure 19f and 19i), providing a bigger exit time T (or a transient state closer to the steady state). In our tests we used a small (but not too small) value of dt because we believe that the first drift-diffusion steps are the most important to propagate the data in and out the shadow domain: in particular, FA-BiCGStab will adapt to the current situation improving the computation time and the accuracy of the expected result. Conversely, a too small starting value of dt is deprecable because the difference of two contiguous solution could be very small satisfying the stopping condition before than expected (even if this condition is weighted by the current dt).

5.7 Applications

Shadow Removal

The ability of the drift-diffusion Equation (5.1) to be invariant under multiplicative grayscale allows us to model shadows as a local multiplicative illumination change within the image. This affects only the canonical drift vectors at the shadows boundaries, the transition locus between the shadows and the non shadows. To complete this task we need as input the original image with a shadow boundary marked in a separate layer as in Figure 20b. Recovering the shadow edges requires other techniques and it is not an argument of this work.

(a) Original image with shadow.



(b) Shadow boundaries.

Figure 20: Data to be known for Shadow Removal.

We can take great advantages from the *preservation of average gray value* (or color value in each channel), which we have seen to be a key property in order to preserve all information stored in the input image. However, once reached the final step desired, we note that the output image is darker than the one expected: roughly speaking, this phenomenon can be understood as if every pixel outside the shadow domain absorbs the a part of the shadow coefficient. For this reason one could adjust the output image to match the data outside the shadow region with the same data of the original image.

(a) Original image with shadow.

(b) Reference with `expmv`.

(c) LUpq, $\theta = 1$.

(d) A-BiCGStab + ilu, $\theta = 0.5$.

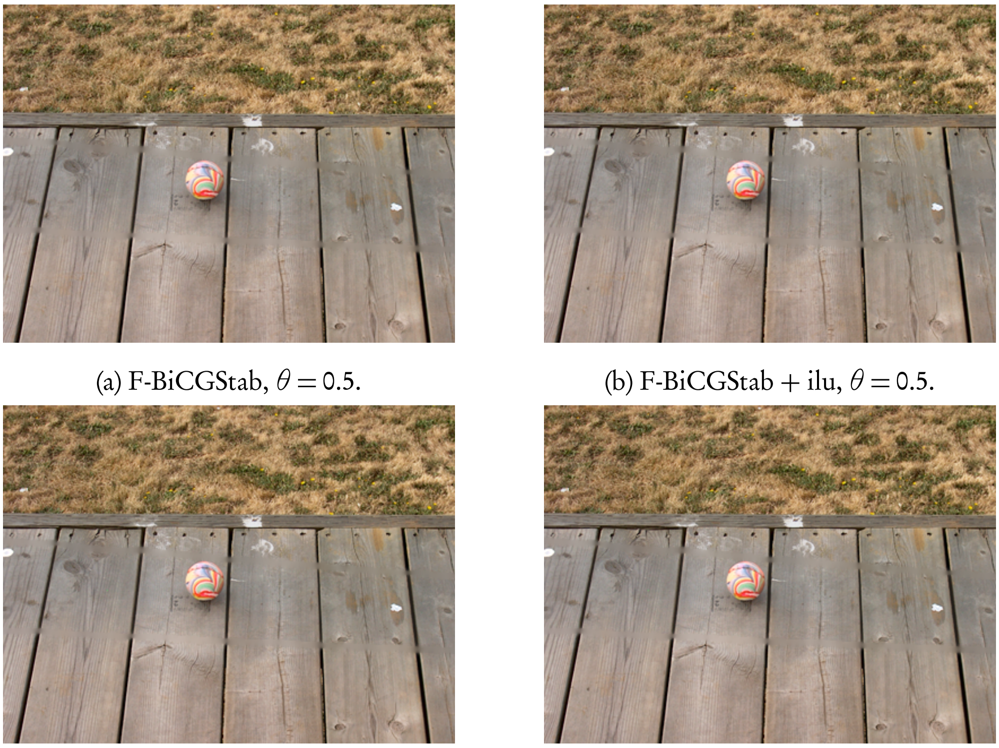
(e) Fourier Alg. 3.

(f) Fourier Alg. 4.

Figure 21: Results from Table 5.3 at $T = 1000$. Fourier results from Figure 18. We observe the visually equivalence of the results. The output is darker than the original because of the diffusion combined with average gray preservation property of the PDE. A global multiplicative factor can be recovered comparing non-shadows area in input and output figures.

Remark 5.10. Only diffusion, i.e. the laplacian operator, acts on boundary inpainting domain so a smooth effect is clearly visible: this can be removed with any of inpainting techniques of Chapter 4.

Remark 5.11. The shadow boundaries detection has been exploited in various researches through last years, for example in [Finlayson *et al.*(2004), Finlayson *et al.*(2006), Finlayson *et al.*(2009)]: these papers are based on the projection of the geometric-mean chromaticity, defined as $C_i / \sqrt[3]{R \cdot G \cdot B}$ with $C_i = R, G$ or B, of an RGB image onto a subspace of $\mathbb{R}^3$. The minimum entropy of these projections



(a) F-BiCGStab, $\theta = 0.5$. (b) F-BiCGStab + ilu, $\theta = 0.5$.

(c) FA-BiCGStab, $\theta = 0.5$. (d) FA-BiCGStab + ilu, $\theta = 0.5$.

Figure 22: Shadow Removal example. Results from Table 5.3 when the exit time condition is fulfilled.

provides an invariant chromaticity. This will be then compared with the original chromaticity to compute the edge strength. So the key idea is to separate the edge in two categories: edges appearing in both chromaticity images (the real edges on the scene) and edges appearing only in the original chromaticity but not in the invariant one (the real shadow edges). Unfortunately, according to the authors of this method, an acceptable result can be obtained with an uncompressed input image, very similar to the RAW image captured by the camera sensor. Other methods are based on comparison between the same scene under flash and no-flash illumination [Drew *et al.*(2006)]. For these reasons, we assume to know the shadow boundaries to be used in our drift-diffusion equation as the place where to impose a zero drift vector field.

Seamless image cloning

Osmosis filter can also be used to fuse incompatible information in *seamless image cloning* process. The problem can be addressed as the following: given the image domain Ω and two images f_1 and f_2, we suppose that f_1 and f_2 share the same image domain. We want to merge a portion of f_1 with the data in f_2. In general, it is sufficient to know the domain of f_2 to be cloned in f_1, with the pixels in f_1 to be substituted. We call the image domain to be cloned as Γ and its boundary is given by $\partial\Gamma$. We suppose also to know f_1 and f_2 at $\partial\Gamma$.

(a) Parameters: $\theta = 0.5$, dt $= 1$, `tol_bicgstab`$=10^{-05}$, `tol_exit`$=10^{-06}$ and maxit $= 30$.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	8349	7415.79	1000	1000	1804	7915.44
I	-	2684.5	948	13447.5	2948	1516.5	342	2432.5	972
R	-	0	0	0	4	0	0	0	0
E	1.47e-08	6.71e-03	1.30e-03	-	-	9.82e-05	1.24e-03	-	-
C	86.80	139.00	17.88	988.69	54.38	191.49	14.41	319.72	34.48

(b) Parameters: $\theta = 1$, dt $= 1$, `tol_bicgstab`$=10^{-05}$, `tol_exit`$=10^{-06}$ and maxit $= 30$.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	2090	7999.14	1000	1000	1805	8247.86
I	-	2924.5	1092.5	4997.5	2837	1531.5	494.5	2449.5	1396
R	-	0	1	0	13	0	0	0	1
E	4.59e-05	4.47e-03	7.58e-03	-	-	1.93e-04	4.09e-03	-	-
C	84.27	118.93	20.08	230.59	56.14	175.20	18.40	292.16	47.36

(c) Comparison between best cputimes (from Table 5.1a and 5.1b at $T = 1000$ with dt $= 1$) of LUpq and BiCGStab with `expv` and Fourier (F.) (Alg. 3 and Alg. 4). For Fourier experiments we choose dt $= 100$, according to experiments in Figure 18.

	Ref. `expmv`	**LUpq**	**BiCGStab**	**BiCGStab + ilu**	`expv`	**F. Alg. 3**	**F. Alg. 4**
T	1000	1000	1000	1000	1000	1000	1000
θ	-	1	0.5	0.5	-	-	-
I	-	-	948	342	-	-	-
R	-	-	0	0	-	-	-
E	-	4.59e-05	1.30e-03	1.24e-03	1.73e-04	0.1275	0.1080
C	206.26	84.27	17.88	14.41	25.07	4.80	10.94

Table 5.1: Shadow Removal results for Figure 21. Results in red are visually wrong (e.g. imperfections across shadow boundaries or output in F-BiCGStab or FA-BiCGStab far away from the expected solution due to exit condition). Legend: **I**=Iterations, **R**=Refused steps, **E**=*Error* from reference solution (or, better, the difference from the reference solution), **C**=Cputime.

(a) Parameters: $\theta = 0.5$, $\mathrm{dt} = 1$, `tol_bicgstab`$=10^{-05}$, `tol_exit`$=10^{-07}$ and maxit $= 30$.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	8349	24686.12	1000	1000	1804	24741.28
I	-	2684.5	948	13447.5	6664.5	1516.5	342	2432.5	1438.5
R	-	0	0	0	6	0	0	0	0
E	1.47e-08	6.71e-03	1.30e-03	-	-	9.82e-05	1.24e-03	-	-
C	86.80	136.74	17.97	996.30	119.62	194.68	14.48	321.95	49.26

(b) Parameters: $\theta = 1$, $\mathrm{dt} = 1$, `tol_bicgstab`$=10^{-05}$, `tol_exit`$=10^{-07}$ and maxit $= 30$.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	2090	23082.62	1000	1000	1805	23520.25
I	-	2924.5	1092.5	4997.5	4072	1531.5	494.5	2449.5	2036
R	-	0	1	0	20	0	0	0	1
E	4.59e-05	4.47e-03	7.58e-03	-	-	1.93e-04	4.07e-03	-	-
C	84.27	120.02	20.25	231.70	81.21	175.94	18.41	295.87	67.42

(c) Comparison between best cputimes (from Table 5.2a and 5.2b at $T = 1000$ with $\mathrm{dt} = 1$) of LUpq and BiCGStab with `expv` and Fourier (F.) (Alg. 3 and Alg. 4). For Fourier experiments we choose $\mathrm{dt} = 100$, according to experiments in Figure 18.

	Ref. `expmv`	**LUpq**	**BiCGStab**	**BiCGStab + ilu**	`expv`	**F. Alg. 3**	**F. Alg. 4**
T	1000	1000	1000	1000	1000	1000	1000
θ	-	1	0.5	0.5	-	-	-
I	-	-	948	342	-	-	-
R	-	-	0	0	-	-	-
E	-	4.59e-05	1.30e-03	1.24e-03	1.73e-04	0.1275	0.1080
C	206.26	84.27	17.97	14.48	25.07	4.80	10.94

Table 5.2: Shadow Removal results for Figure 21. Results in red are visually wrong (e.g. imperfections across shadow boundaries or output in F-BiCGStab or FA-BiCGStab far away from the expected solution due to exit condition). Legend: **I**=Iterations, **R**=Refused steps, **E**=*Error* from reference solution (or, better, the difference from the reference solution), **C**=Cputime.

(a) Parameters: $\theta = 0.5$, $\mathrm{dt} = 1$, `tol_bicgstab`$=10^{-06}$, `tol_exit`$=10^{-06}$ and $\mathrm{maxit} = 30$.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	5963	7177.26	1000	1000	6933	7579.16
I	-	2968.5	1359	15394	4082	1585	516	9979.5	1485
R	-	0	0	0	4	0	0	0	2
E	1.47e-08	1.68e-04	1.19e-03	-	-	1.03e-04	1.21e-03	-	-
C	86.80	138.12	24.43	804.32	73.34	192.73	19.37	1308.37	51.36

(b) Parameters: $\theta = 1$, $\mathrm{dt} = 1$, `tol_bicgstab`$=10^{-06}$, `tol_exit`$=10^{-06}$ and $\mathrm{maxit} = 30$.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	5618	7798.39	1000	1000	6935	7856.53
I	-	3306.5	1717	15897.5	4529.5	1785	753	10180.5	2298.5
R	-	0	1	0	13	0	0	0	2
E	4.59e-05	1.47e-04	4.53e-03	-	-	2.04e-04	3.94e-03	-	-
C	84.27	125.36	30.04	679.08	83.84	181.92	25.63	1203.16	75.19

(c) Comparison between best cputimes (from Table 5.3a and 5.3b at $T = 1000$ with $\mathrm{dt} = 1$) of LUpq and BiCGStab with `expv` and Fourier (F.) (Alg. 3 and Alg. 4). For Fourier experiments we choose $\mathrm{dt} = 100$, according to experiments in Figure 18.

	Ref. `expmv`	**LUpq**	**BiCGStab**	**BiCGStab + ilu**	`expv`	**F. Alg. 3**	**F. Alg. 4**
T	1000	1000	1000	1000	1000	1000	1000
θ	-	1	0.5	0.5	-	-	-
I	-	-	1359	516	-	-	-
R	-	-	0	0	-	-	-
E	-	4.59e-05	1.19e-03	1.21e-03	1.73e-04	0.1275	0.1080
C	206.26	84.27	24.43	19.37	25.07	4.80	10.94

Table 5.3: Shadow Removal results for Figure 21. Results in red are visually wrong (e.g. imperfections across shadow boundaries or output in F-BiCGStab or FA-BiCGStab far away from the expected solution due to exit condition). Legend: **I**=Iterations, **R**=Refused steps, **E**=*Error* from reference solution (or, better, the difference from the reference solution), **C**=Cputime.

(a) Parameters: $\theta = 0.5$, dt $= 1$, `tol_bicgstab`$=10^{-06}$, `tol_exit`$=10^{-07}$ and maxit $= 30$.

	LUpq	BiCGStab				BiCGStab + ilu			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	16982	22945.86	1000	1000	6934	23153.42
I	-	2968.5	1359	31675	7215.5	1585	516	9979.5	2311
R	-	0	0	0	11	0	0	0	2
E	1.47e-08	1.68e-04	1.19e-03	-	-	1.03e-04	1.21e-03	-	-
C	86.80	139.48	24.63	2094.52	131.96	194.28	19.5	1313.19	77.4

(b) Parameters: $\theta = 1$, dt $= 1$, `tol_bicgstab`$=10^{-06}$, `tol_exit`$=10^{-07}$ and maxit $= 30$.

	LUpq	BiCGStab				BiCGStab + ilu			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	7506	22761.78	1000	1000	6935	23162.34
I	-	3306.5	1717	20274.5	7111	1785	753	10180.5	3498.5
R	-	0	1	0	23	0	0	0	6
E	4.59e-05	1.47e-04	4.53e-03	-	-	2.04e-04	3.94e-03	-	-
C	84.27	126.64	30.45	897.3	134.36	182.08	25.88	1206.13	116.62

(c) Comparison between best cputimes (from Table 5.4a and 5.4b at $T = 1000$ with dt $= 1$) of LUpq and BiCGStab with `expv` and Fourier (F.) (Alg. 3 and Alg. 4). For Fourier experiments we choose dt $= 100$, according to experiments in Figure 18.

	Ref. `expmv`	**LUpq**	**BiCGStab**	**BiCGStab + ilu**	`expv`	**F. Alg. 3**	**F. Alg. 4**
T	1000	1000	1000	1000	1000	1000	1000
θ	-	1	0.5	0.5	-	-	-
I	-	-	1359	516	-	-	-
R	-	-	0	0	-	-	-
E	-	4.59e-05	1.19e-03	1.21e-03	1.73e-04	0.1275	0.1080
C	206.26	84.27	24.63	19.5	25.07	4.80	10.94

Table 5.4: Shadow Removal results for Figure 21. Results in red are visually wrong (e.g. imperfections across shadow boundaries or output in F-BiCGStab or FA-BiCGStab far away from the expected solution due to exit condition). Legend: **I**=Iterations, **R**=Refused steps, **E**=*Error* from reference solution (or, better, the difference from the reference solution), **C**=Cputime.

(a) Parameters: $\theta = 0.5$, $\mathrm{dt} = 1$, `tol_bicgstab`$=10^{-07}$, `tol_exit`$=10^{-06}$ and maxit $= 30$.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	6868	7464.87	1000	1000	6933	7341.04
I	-	3398.5	1872	18621.5	5992.5	3066	680.5	12295	2233
R	-	0	4	0	15	0	0	0	1
E	1.47e-08	2.24e-05	4.08e-03	-	-	9.58e-06	1.20e-03	-	-
C	86.80	152.28	35.32	949.07	112.82	234.71	24.36	1383.97	73.92

(b) Parameters: $\theta = 1$, $\mathrm{dt} = 1$, `tol_bicgstab`$=10^{-07}$, `tol_exit`$=10^{-06}$ and maxit $= 30$.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	6867	7674.77	1000	1000	6935	7423.52
I	-	4032.5	2407.5	20732.5	6524.5	3332.5	1036	15644	3576
R	-	0	4	0	27	0	0	0	4
E	4.59e-05	6.52e-05	4.80e-03	-	-	7.21e-05	3.22e-03	-	-
C	84.27	138.65	43.60	874.73	126.36	225.51	34.36	1360.56	116.99

(c) Comparison between best cputimes (from Table 5.5a and 5.5b at $T = 1000$ with $\mathrm{dt} = 1$) of LUpq and BiCGStab with `expv` and Fourier (F.) (Alg. 3 and Alg. 4). For Fourier experiments we choose $\mathrm{dt} = 100$, according to experiments in Figure 18.

	Ref. `expmv`	**LUpq**	**BiCGStab**	**BiCGStab + ilu**	`expv`	**F. Alg. 3**	**F. Alg. 4**
T	1000	1000	1000	1000	1000	1000	1000
θ	-	1	0.5	0.5	-	-	-
I	-	-	1872	680.5	-	-	-
R	-	-	4	0	-	-	-
E	-	4.59e-05	4.08e-03	1.20e-03	1.73e-04	0.1275	0.1080
C	206.26	84.27	35.32	24.36	25.07	4.80	10.94

Table 5.5: Shadow Removal results for Figure 21. Results in red are visually wrong (e.g. imperfections across shadow boundaries or output in F-BiCGStab or FA-BiCGStab far away from the expected solution due to exit condition). Legend: **I**=Iterations, **R**=Refused steps, **E**=*Error* from reference solution (or, better, the difference from the reference solution), **C**=Cputime.

(a) Parameters: $\theta = 0.5$, dt = 1, `tol_bicgstab`$=10^{-07}$, `tol_exit`$=10^{-07}$ and maxit = 30.

	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	16619	23061.54	1000	1000	20430	24228.54
I	-	3398.5	1872	38248	11660	3066	680.5	27837.5	3859
R	-	0	4	0	24	0	0	0	7
E	1.47e-08	2.24e-05	4.09e-03	-	-	9.58e-06	1.20e-03	-	-
C	86.80	147.61	35.2	1996	215.72	233.46	24.33	3428.48	130.35

(b) Parameters: $\theta = 1$, dt = 1, `tol_bicgstab`$=10^{-07}$, `tol_exit`$=10^{-07}$ and maxit = 30.

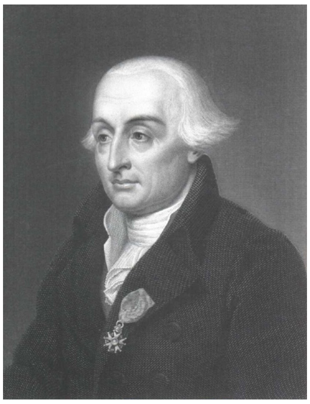
	LUpq	**BiCGStab**				**BiCGStab + ilu**			
		-	A	F	FA	-	A	F	FA
T	1000	1000	1000	15358	22299.95	1000	1000	20433	21338.26
I	-	4032.5	2407.5	39377.5	10806	3332.5	1036	31189	5774.5
R	-	0	4	0	47	0	0	0	6
E	4.59e-05	6.59e-05	4.80e-03	-	-	7.21e-05	3.22e-03	-	-
C	84.27	137.13	43.44	1748.14	209.73	225.12	34.25	3240.42	187.02

(c) Comparison between best cputimes (from Table 5.6a and 5.6b at $T = 1000$ with dt = 1) of LUpq and BiCGStab with `expv` and Fourier (F.) (Alg. 3 and Alg. 4). For Fourier experiments we choose dt = 100, according to experiments in Figure 18.

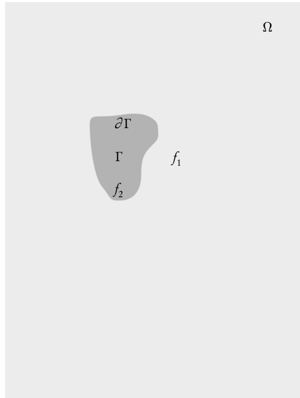
	Ref. `expmv`	**LUpq**	**BiCGStab**	**BiCGStab + ilu**	`expv`	**F. Alg. 3**	**F. Alg. 4**
T	1000	1000	1000	1000	1000	1000	1000
θ	-	1	0.5	0.5	-	-	-
I	-	-	1872	680.5	-	-	-
R	-	-	4	0	-	-	-
E	-	4.59e-05	4.09e-03	1.20e-03	1.73e-04	0.1275	0.1080
C	206.26	84.27	35.2	24.33	25.07	4.80	10.94

Table 5.6: Shadow Removal results for Figure 21. Results in red are visually wrong (e.g. imperfections across shadow boundaries or output in F-BiCGStab or FA-BiCGStab far away from the expected solution due to exit condition). Legend: **I**=Iterations, **R**=Refused steps, **E**=*Error* from reference solution (or, better, the difference from the reference solution), **C**=Cputime.

(a) Euler.



(b) Lagrange.



(c) Domain notation.

(d) Input.

(e) [Pérez *et al.*(2003)].

(f) Osmosis filter.

Figure 23: Seamless image cloning. Figure 23e from [Weickert *et al.*(2013)] differs from our test for the cloning domain but the visual expectation of more natural transient zone between f_1 and f_2 in Figure 23f is confirmed as expected. Source of original images: Wikimedia Commons.

The most famous attempt to solve this problem is *Poisson image editing* in [Pérez *et al.*(2003)]. Referred to notation in Figure 23c, the simplest interpolant f_2 of f_1 over Γ is the membrane interpolant defined as the solution of the minimization problem:

$$\min_{f_2} \int\int_{\Gamma} |\nabla f_2|^2 \quad \text{with} \quad f_2|_{\partial\Gamma} = f_1|_{\partial\Gamma}. \tag{5.8}$$

The minimizer must satisfy the associated Euler-Lagrange equation

$$\begin{cases} \Delta f_2 = 0, & \text{on } \Gamma, \\ f_2 = f_1, & \text{on } \partial\Gamma. \end{cases}$$

However, this method produces an unsatisfactory, blurred interpolant. For this reason, a *guidance vector field* $\mathbf{p}$ is introduced in an extended version of the minimization problem (5.8):

$$\min_{f_2} \int\int_{\Gamma} |\nabla f_2 - \mathbf{p}| \quad \text{with} \quad f_2|_{\partial\Gamma} = f_1|_{\partial\Gamma},$$

whose solution is the unique solution of the following Poisson equation with Dirichlet boundary conditions:

$$\begin{cases} \Delta f_2 = \operatorname{div} \mathbf{p}, & \text{on } \Gamma \\ f_2 = f_1, & \text{on } \partial\Gamma, \end{cases} \tag{5.9}$$

where $\operatorname{div} \mathbf{p} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$ is the divergence of $\mathbf{p} = (u, v)$. This is the fundamental machinery of Poisson editing of color images: three Poisson equations of the form (5.9) are solved independently in the three color channels of the chosen color space. When the guidance field $\mathbf{p}$ is conservative, i.e. it is the gradient of some function g, a helpful alternative way of understanding what Poisson interpolation does is to define the correction function $\widetilde{f_2}$ on Γ such that $f_2 = g + \widetilde{f_2}$. The Poisson equation (5.9) then becomes the following Laplace equation with boundary conditions:

$$\begin{cases} \Delta \widetilde{f_2} = 0, & \text{on } \Gamma, \\ \widetilde{f_2} = (f_1 - g), & \text{on } \partial\Gamma. \end{cases}$$

Therefore, inside Γ, the additive correction $\widetilde{f}$ is a membrane interpolant of the mismatch $(f_1 - g)$ between the source and the destination along the boundary $\partial\Gamma$.

So, imposing $\mathbf{p} = \nabla f_2$ in Γ and Dirichlet boundary conditions $f_2 = f_1$ on $\partial\Gamma$ in Equation (5.9), we can obtain a nice image but this method it is not able to match the different illumination of f_1 and f_2. Now, to provide a seamless osmotic cloning using the drift-diffusion model proposed in Equation (5.1), we use the canonical drift vectors on f_1 in $\Omega \setminus \Gamma$ and the ones in f_2 in Γ while, at the interface $\partial\Gamma$, we use the arithmetic mean of both drift vectors: this returns smoother results than [Pérez *et al.*(2003)] as we can see in Figure 23.

CHAPTER 6

Conclusion and future works

We showed a connection between NL-Poisson inpainting approach by [Arias *et al.*(2011)] and the drift-diffusion equation by [Weickert *et al.*(2013)]. We deepened these two arguments in separated topics: the former recalls some definitions and useful properties of BV space (the appropriate set where to model most of the problems in Computer Vision) with a presentation of different approaches based on curvature and patch comparison for the inpainting problem; the latter with some numerical experiments that introduce fast methods in solving the drift-diffusion equation. In particular, we state that BiCGStab method, or its variants, needs a small initial timestep for converging to the steady state when the reference solution is unknown (this is the most common case for the applications we have in mind), but this is not the best choice yet to speed up the cputime: it may be more convenient to use a Fourier approach with a fixed big final time (or with a stopping criterion similar to that one used for BiCGStab experiments) where we expect that the steady state is reached, accepting a small, but still undetectable at our eyes, error based on Gibbs phenomenon or based on separation of the drift step from the diffusion step.

From our experiments, clearly arise that relaxing conditions on expected accuracy don't compromise at all the visual steady state result: this could be acceptable for most of the users with the advantage of a terrific speed up in computations, especially when using Fourier methods with large timesteps. However, where a better control on solution accuracy is needed, we proposed some BiCGStab variants to adapt the timesteps used to the iterations of BiCGStab.

As future works we expect to provide a better control on stopping criterion for BiCGStab iterations, in order to prevent false steady state convergence. Moreover, the problem in recognizing the shadow area is not solved at all, especially when the shadow is not a costant. For this reasons, we expect that the use of an exemplar-based variational model could *inpaint* the shadow area not with the creation of an admissible texture missed, but restoring the true lighting coefficient.

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